Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 9, Number 1 (2013), pp. 13-24

http://www.ripublication.com/gjpam.htm

Classification of Operators with the Property $\operatorname{Re}\sigma(T) = \sigma(\operatorname{Re} T)$

[1]Were Jamen H. and [2]Mile Justus K.

University of Nairobi, School of Mathematics,
P.O. Box 30197-00100,Nairobi, Kenya.
EMAIL : milejustus@yahoo.com, Telephone : +254721933909.

ABSTRACT

In this paper, we are motivated towards a natural problem to find classes of non-normal operators which satisfy one of the most important property of the spectrum that $\operatorname{Re}\sigma(T) = \sigma(\operatorname{Re} T)$.

Simple examples show that this remarkable relation does not hold for arbitrary operators. It is clear that for nilpotent operators or more generally for quasinilpotent operators, this relation fails to hold.
However, if T is bounded operator and one of the following conditions holds, then the relation $\operatorname{Re}\sigma(T) = \sigma(\operatorname{Re} T)$ is true:

1. T is hyponormal
2. The adjoint of T, T^*, is hyponormal
3. Spectrum of T, $\sigma(T)$, is a spectral set of T
4. T has the G_1-property and spectrum of T is connected.

LIST OF NOTATIONS

$\operatorname{Re}\sigma(T)$	real part of Spectrum T
$\rho(T)$	resolvent set of T
$Conv\sigma(T)$	convex hull of spectrum of T
$P_\sigma(T)$	point spectrum of T
$\pi(T)$	approximate spectrum of T
Ker(T)	kernel of T
$\overline{W(T)}$	closure of numerical range of T

T^*	adjoint of T
$\mathbb{C}$	complex plane

INTRODUCTION

The theory of normal operators is so successful that much of the theory of non-normal operators is modeled after it.

If T is a normal operator in B(H), the set of all bounded operators on a Hilbert space H, then it has a number of interesting properties, among which are the following

(i) $r(T) = \|T\|$

(ii) $\text{Conv}\,\sigma(T) = \overline{W(T)}$

(iii) $\|Tx\| = \|T^*x\|$ for all $x \in H$

(iv) For all $\lambda \in \mathbb{C}$, $T + \lambda I$ is a normal operator.

(v) If M is an invariant subspace of T, then the restriction $T|_M$ has property (i).

(vi) If M is a reducing subspace of T, i.e, M is invariant under T and T*, then $T|_M$ is normal.

(vii) $\sigma(T)$ and $\overline{W(T)}$ are spectral sets of T.

(viii) $r(T) = w(T)$, the numerical radius of T.

(ix) T satisfies the G_1-property, that is,

$$\|(T - \lambda I)^{-1}\| = \frac{1}{dist(\lambda; \sigma(T))}$$

for all $\lambda \in \rho(T)$.

(x) $\text{Re}\,\sigma(T) = \sigma(\text{Re}\,T)$.

Historically, the consideration of non-normal operators began with a famous paper of Wintner (1929) in an attempt to obtain a characterization of operators T for which $r(T) = \|T\|$. He asserted that for this property it is necessary and sufficient that $Conv\,\sigma(T) = \overline{W(T)}$. An example of Paul R. Halmos (1967) shows that this is not the case.

Thus properties (i) and (ii) are not equivalent.

Our aim is to consider weaker operators which satisfy some of the properties that normal operators fulfill or their weaker versions and find the interconnections, if any, between some of these classes of non-normal operators.

In this paper, we characterize those operators $T \in B(H)$ for which $\mathrm{Re}\,\sigma(T) = \sigma(\mathrm{Re}\,T)$ generally.

In particular, we show that: If $T \in B(H)$ and one of the following holds, then T has the property $\mathrm{Re}\,\sigma(T) = \sigma(\mathrm{Re}\,T)$:

(i) T is hyponormal.

(ii) T* is hyponormal

(iii) $\sigma(T)$ is a spectral set

(iv) T has G_1-property and $\sigma(T)$ is connected.

DEFINITIONS AND CONSEQUENCES

Definition 1

A closed subset X of the complex plane is called a spectral set for an operator

$T \in B(H)$ if

(i) $\sigma(T) \subseteq X$ and

(ii) for all $f \in R(X)$

$\| f(T) \| \le \| f \|_X \left(= \sup\{ |f(z)| : z \in X \} \right)$ (see Fillmore[4]pg.62)

For normal operators $T \in B(H)$, it is known that $\sigma(T)$ and $\overline{W(T)}$ are spectral sets.

(see Halmos[6]pg122)

Definition 2

A operator $T \in B(H)$ is said to have G_1-property if

$$\left\| (T - \lambda I)^{-1} \right\| = \frac{1}{dist(\lambda, \sigma(T))} \text{ for all } \lambda \in \rho(T).$$

When $T \in B(H)$ is hyponormal, it is clear that $T - \lambda I$ is also hyponormal for all $\lambda \in \mathbb{C}$ and if, in addition, T is invertible, then T^{-1} is also hyponormal. (see Fillmore [4]).

Definition 3

An operator $T \in B(H)$ is said to be in spectral G_1-class (or with spectral G_1-property or equivalently, a convexoid operator) if for all $z \notin conv\,\sigma(T)$

$$\left\|(T-zI)^{-1}\right\|,\ i.e.,\ \ \|R_z(T)\| \le \frac{1}{dist(z, Conv\ \sigma(T)).}$$

The basic result due to S. Berberian is the following:

Proposition 1
Let $T \in B(H)$. Then there exist a Hilbert space $\tilde{H}$, and an application $T \mapsto \tilde{T}$ such that $W(\tilde{T}) = \overline{W(T)}$ and $P_\sigma(\tilde{T}) = \pi(T)$. (see Berberian and Orland [2])

Definition 4
An operator $T \in B(H)$ is called Translation-invariant –normaloid if $T_\lambda = T + \lambda I$ satisfies $\|T_\lambda\| = r_{T_\lambda}$ for all $\lambda \in \mathbb{C}$.

Proposition 2
If $T \in B(H)$ is Translation- invariant-normaloid, then

$$\overline{W(T)} = \text{Conv}\,\sigma(T)$$

We first prove the following results:

Proposition 3
If $T \in B(H)$ is a hyponormal operator, then $\text{Re}\,\pi(T) \subset \sigma(\text{Re}\,T)$

Proof
Let $\lambda = x + iy \in \pi(T)$. Then there exists a sequence (x_n) of elements of H such that $\|x_n\| = 1$ and $(T - \lambda I)x_n \xrightarrow{s} \overline{0}$.

Since $T - \lambda I$ is hyponormal
$(T^* - \overline{\lambda} I)x_n \xrightarrow{s} \overline{0}.$

Hence

$$(\text{Re}\,T)x_n - \text{Re}\,\lambda x_n = \frac{1}{2}\left[T + T^* - (\lambda + \overline{\lambda})I\right]x_n \xrightarrow{s} \overline{0}$$

and this proves that

$$\text{Re}\,\lambda \in \pi\left(\frac{T + T^*}{2}\right) = \sigma(\text{Re}\,T)$$ for Re T is self-adjoint. Thus the result.

Proposition 4
If $T \in B(H)$ is hyponormal, then $\text{Re}\,\sigma(T) \subset \sigma(\text{Re}\,T)$.

Proof

Let $\lambda_o \in \sigma(T)$ and the line $\operatorname{Re}\lambda = \operatorname{Re}\lambda_o$ meet the spectrum of T in a boundary point λ_0'. Since every boundary point of $\sigma(T)$ is in the approximate point spectrum $\pi(T)$, it follows that $\operatorname{Re}\lambda_o'$ is in $\operatorname{Re}\pi(T)$ and hence in $\sigma(\operatorname{Re}T)$ (by Proposition 3).

But $\operatorname{Re}\lambda_o = \operatorname{Re}\lambda_0'$, and we have thus proved the assertion of the proposition.

Remark:

Clearly, the assertion of proposition 4 also holds when T* is hyponormal.

Proposition 5

If $T \in B(H)$ and if either T or T* is hyponormal, then $\operatorname{Re}\sigma(T)=\sigma(\operatorname{Re}T)$.

Proof

We need only consider the case when T is hyponormal. We use the result that the approximate point spectrum of a self-adjoint operator is exactly its spectrum. Using proposition 1, we can assume, without loss of generality, that if $T = A+iB$, then $\sigma(A)=P_\sigma(A)$.

Let $\lambda = a+ib$ and $a\in\sigma(A)$. Consider the subspace $M=\ker(A-aI)$.

Since a is an eigenvalue, $M \neq \{\overline{0}\}$. We show that M is an invariant subspace for B. Let $x\in M$ and thus $Ax = ax$.

Since we have

$$(A-aI)B - B(A-aI) = \frac{1}{2}iD \quad \text{where} \quad D = T^*T - TT^*, \text{ it follows that}$$

$$-\frac{1}{2}i\langle Dx,x\rangle = \langle (A-aI)Bx,x\rangle - \langle B(A-aI)x,x\rangle = 0$$

and thus $\langle Dx,x\rangle = 0.$

But $D>0$ and thus $D^{\frac{1}{2}}$ exists and is self-adjoint. Hence we have

$$0=\langle Dx,x\rangle = \langle D^{\frac{1}{2}}x, D^{\frac{1}{2}}x\rangle = \left\|D^{\frac{1}{2}}x\right\|^2 \text{ which implies that } D^{\frac{1}{2}}x=\overline{0}.$$

From this we have $Dx=\overline{0}$ and hence

$$0 = A(Bx) - B(Ax) = (A-aI)Bx = \overline{0} \text{ that is, } Bx \in M.$$

Clearly M is an invariant subspace for T and $T|_M$ is of the form $\tilde{T} = aI + \tilde{B}$ and since $\tilde{B}$ is hermitian (self-adjoint), $\tilde{T}$ is a normal operator.

But T is a hyponormal operator whose restriction to an invariant subspace is normal.

Then M is also invariant for T*. Clearly,

$$T = \left(T|_M\right) \oplus (T|_{M^\perp}) = T_1 \oplus T_2.$$

Since $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, we obtain that the same relation holds for the real parts,

$$\operatorname{Re}\sigma(T) = \operatorname{Re}\sigma(T_1) \cup \operatorname{Re}\sigma(T_2)$$

and since $\operatorname{Re}\sigma(T_1) = \{a\}$, the assertion of the proposition follows.

Proposition 6
If $T \in B(H)$ is a convexoid operator and $[a,b]$ is the smallest segment containing $\operatorname{Re}\sigma(T)$, then a and b are in $\sigma(\operatorname{Re} T)$.

Proof
As we know, we can assume without loss of generality that W(T) is a closed convex set and $\pi(T) = P_\sigma(T)$ (Proposition 1).

Let $\lambda_o \in \sigma(T)$ such that $\operatorname{Re}\lambda_o = a$ and λ_o is a point on $\partial\sigma(T)$ such that $\lambda_o \in W(T)$.

Since T is convexoid, W(T) is closed and $\operatorname{Re}\sigma(T) \geq a$, it follows that

$$\operatorname{Re}\overline{W(T)} = \operatorname{Re}(conv\sigma(T)) \geq a$$

and thus λ_o is a boundary point of W(T), ie, $\lambda_o \in \partial W(T)$.

Hence, as is easily seen, λ_o is a normal eigenvalue.

As in proposition 3 we can show that a is in $\sigma(\operatorname{Re} T)$, and similarly for b.

Proposition 7
If $T \in B(H)$ is a convexoid operator and [c,d] is the smallest segment containing

$\sigma(\operatorname{Re} T)$, then $\sigma(\operatorname{Re} T) \subset [c,d] \subset \operatorname{Re}\sigma(T)$ if $\sigma(T)$ is a connected set.

Proof
Since $\sigma(T)$ is connected, it follows that $\operatorname{Re}\sigma(T)$ is a segment on $\Re$.

To prove the proposition it suffices to show that $c,d \in \operatorname{Re}\sigma(T)$. Let us suppose the contrary for c first, i.e, $c \notin \operatorname{Re}\sigma(T)$.

Let l be the straight line $\mathrm{Re}\,\lambda = c$. l is disjoint from $\sigma(T)$ since otherwise $c \in \mathrm{Re}\,\sigma(T)$.

But $\sigma(T)$ is a connected set and thus it is strictly on one side of l. Suppose that it is on the right side of l. Then we can find an $\varepsilon > 0$ such that $\mathrm{Re}\,\sigma(T) \geq c+\varepsilon$.

Since T is convexoid, we have $conv\sigma(T)=\overline{W(T)}$ so $W(T)\subset conv\sigma(T)$ and hence

$\mathrm{Re}W(T)\geq c+\varepsilon$, i.e, $\mathrm{Re}\langle Tx,x\rangle \geq (c+\varepsilon)\|x\|^2$ for all $x\in H$.

Since $\mathrm{Re}\langle Tx,x\rangle = \langle(\mathrm{Re}\,T)x,x\rangle$ for all $x\in H$, we obtain

$\langle(\mathrm{Re}\,T)x,x\rangle \geq (c+\varepsilon)\|x\|^2$ for all $x\in H$, i.e. $\mathrm{Re}\,T\geq (c+\varepsilon)I$, which implies

$\|(\mathrm{Re}\,T)x - cIx\| \geq \varepsilon\|x\|$ for all $x\in H$.

This shows that

$c\notin \pi(\mathrm{Re}\,T)=\sigma(\mathrm{Re}\,T)$ (for $\mathrm{Re}\,T$ is self –adjoint)

and this contradicts the hypothesis that [c, d] is the smallest segment containing $\sigma(\mathrm{Re}\,T)$. Likewise, dealing with d, we can show that $d\notin\sigma(\mathrm{Re}\,T)$ and this gives again a contradiction of the hypothesis.

Hence $c,d\in \mathrm{Re}\,\sigma(T)$.

Note

[c,d] is the smallest segment containing $\sigma(\mathrm{Re}\,T)$ implies that

$$c=\inf\{\langle(\mathrm{Re}\,T)x,x\rangle : x\in H \text{ and } \|x\|=1\}$$

$$d=\sup\{\langle(\mathrm{Re}\,T)x,x\rangle : x\in H \text{ and } \|x\|=1\}$$

Moreover $c,d\in\sigma(\mathrm{Re}\,T)$.

Proposition 8

If $T\in B(H)$ and $\sigma(T)$ is a spectral set for T, then $\sigma(\mathrm{Re}\,T)\subseteq \mathrm{Re}(\sigma(T))$.

Proof

Since $\sigma(T)$ is a spectral set for T, we have

$\|f(T)\|\leq \sup\{|f(\lambda)| : \lambda\in\sigma(T)\}$ for all rational functions f without poles in $\sigma(T)$.

Now $r_{f(T)} = \sup\{|\lambda| : \lambda\in\sigma(f(T))\}$.

By the spectral mapping theorem

$\sigma(f(T)) = f(\sigma(T))$.

Hence

$r_{f(T)} = \sup\{|\lambda| : \lambda \in f(\sigma(T))\} \quad = \sup\{|f(\lambda)| : \lambda \in \sigma(T)\} \quad \geq \|f(T)\|$.

However $\|f(T)\| \geq r_{f(T)}$ always and hence $r_{f(T)} = \|f(T)\|$.

This shows that f(T) is normaloid.

Taking $f(\lambda) = \lambda - z$ (where z is a constant), we see that $T - zI$ is normaloid, and this says that T is transaloid. Therefore (by proposition 2) we have $conv\sigma(T) = \overline{W(T)}$.

Let $a \in \sigma(\mathrm{Re}\,T)$ and suppose that $a \notin \mathrm{Re}\,\sigma(T)$. Let l be the straight line $\mathrm{Re}\,\lambda = a$ which is disjoint from $\sigma(T)$. Suppose that $\sigma(T)$ is on the left side of l. Then there exists an $\varepsilon > 0$ such that $\mathrm{Re}\,\sigma(T) \leq a - \varepsilon$. But T is convexoid and hence we obtain that

$\mathrm{Re}\,W(T) \leq a - \varepsilon$, i.e., $W(\mathrm{Re}\,T) \leq a - \varepsilon$ .Thus $\sigma(\mathrm{Re}\,T) \leq a - \varepsilon$, which is a contradiction since $a \leq a - \varepsilon$ is not possible.

Incase $\sigma(T)$ is on the right side of l, we can proceed in the same manner.

If $\sigma(T) = A_1 \cup A_2$,where A_1 is on the left of l and A_2 on the right side, then we can decompose $T = T_1 \oplus T_2, \sigma(T_i) = A_i, i = 1,2$ and we can apply the above result.

Proposition 9

If $T \in B(H)$ has the G_1-property,

then $\mathrm{Re}(\sigma(T)) \subset \sigma(\mathrm{Re}\,T)$,

Proof

Assume, without loss of generality that $\pi(T) = P_\sigma(T)$ (see Berberian's result).

This implies that

$\partial\sigma(T) \subset \pi(T) = P_\sigma(T)$.

Let $a \in \mathrm{Re}\,\sigma(T)$ and l be the line $\mathrm{Re}\,\lambda = a$.

Let λ_0 be the point where l exits the spectrum of T; it is clear that it is a boundary point.

Obviously $\operatorname{Re}\lambda_0 = a$.

We can construct a sequence of unit vectors(x_n) such that

(i) $(T-\lambda_0 I)x_n \to \overline{0}$

(ii) $(T^* - \lambda_0^* I)x_n \to \overline{0}$

and this implies that $a \in \sigma(\operatorname{Re}T)$.

For $n = 1,2,3,...,$ let $D_n = \left\{\lambda : |\lambda - \lambda_0| \le \frac{1}{n}\right\}$.

Since $\lambda_0 \in \sigma(T)$, the set D_n contains a point $a_n \in \rho(T)$ such that $|a_n - \lambda_0| \le \frac{1}{2n}$.

Let b_n be such that

$$dist(a_n, \sigma(T)) = |a_n - b_n|.$$

In this case b_n is in $\sigma(T)$, and since T has G_1-property, we obtain

$$\eta_{T-b_n I} = \eta_{T^* - b_n^* I}$$

Now let x_n be such that

$$(T - b_n I)x_n = (T^* - b_n^* I)x_n = \overline{0}$$

and from the definition of b_n we obtain that

$$(T - \lambda_o I)x_n = (b_n - \lambda_o)x_n$$

and

$$(T^* - \lambda_o^* I)x_n = (b_n^* - \lambda_0^* I)x_n.$$

Thus since (x_n) is a sequence satisfying the requirements and the proposition is proved.

Theorem 1

If $T \in B(H)$ has one of the following properties

(i) $\sigma(T)$ is a spectral set for T

(ii) T has the G_1-property and $\sigma(T)$ is connected,

then T has the property $\operatorname{Re}\sigma(T) = \sigma(\operatorname{Re}T)$.

Proof

First, if $\sigma(T)$ is a spectral set, then by proposition 8

$$\sigma(\operatorname{Re} T) \subset \operatorname{Re} \sigma(T).$$

Since T satisfies trivially the G_1- property (by the fact that $\sigma(T)$ is a spectral set), the opposite inclusion follows from proposition 9.

Now suppose that T has G_1-property and $\sigma(T)$ is connected. From proposition 9 we have one inclusion.

For the opposite inclusion, we proceed as follows:

Let [a,b] be the smallest segment containing $\sigma(\operatorname{Re} T)$ and since T is convexoid, by proposition 6, it follows that $[a,b] \subset \operatorname{Re} \sigma(T)$.

Thus we have

$$\sigma(\operatorname{Re} T) \subset [a,b] \subset \operatorname{Re} \sigma(T) \subset \sigma(\operatorname{Re} T)$$

and the second part follows.

Combining theorem1 with proposition 5, we summarize our main result as:

Theorem 2

If $T \in B(H)$ and one of the following conditions holds, then T has the property $\operatorname{Re} \sigma(T) = \sigma(\operatorname{Re} T)$

(i) T is hyponormal

(ii) T^* is hyponormal

(iii) $\sigma(T)$ is a spectral set for T

T has the G_1-property and $\sigma(T)$ is connected.

Example

Now part (iv) of theorem2 does not generalize to arbitrary convexoid operators.

Let A be an operator with matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and B with matrix

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

where the eigenvalues lie off the imaginary axis and are vertices of a triangle containing the disc

$$D_{\frac{1}{2}} = \left\{ z : |z| \le \frac{1}{2} \right\}.$$

Since $W(A) = D_{\frac{1}{2}}$ and $W(B) = conv\{a_1, a_2, a_3\} \supset D_{\frac{1}{2}}$, it follows that $T = A \oplus B$ is convexoid and since the spectrum of $A \oplus B$ is $\{0, a_1, a_2, a_3\}$, it follows that

$$0 \in \operatorname{Re}\sigma(A \oplus B).$$

Since

$$\operatorname{Re}(A \oplus B) = \operatorname{Re}A \oplus \operatorname{Re}B,$$

$$\sigma(\operatorname{Re}(A \oplus B)) = \sigma(\operatorname{Re}A) \cup \sigma(\operatorname{Re}B)$$

$$= \left\{ -\frac{1}{2}, \frac{1}{2} \right\} \cup \operatorname{Re}\sigma(B)$$

and thus

$$0 \notin \sigma(\operatorname{Re}(A \oplus B)).$$

CONCLUSIONS AND RECOMMENDATIONS

As we have seen, a normal operator satisfies quite a number of conditions weaker than normality, for example, hyponormality, convexoidity, transloidity.

Results in the converse direction can be obtained by supplementing or strengthening such conditions.

Quite possibly, the supplementary conditions may refer to the spectrum, finite- dimensionality, compactness, etc. The work on non-normal operators may be pursued in this direction.

REFERENCES

[1] G. Bachman and L. Narici, Functional Analysis, Dover Publications,2000.

[2] S.K. Berbarian and G.H. Orland, 1967. On the Closure of the Numerical Range of an Operator, Proc. Amer. Math. Soc. 18(1967).

[3] F.F. Bonsall and J. Duncan, Studies in Functional Analysis, Amer. Math. Soc. 21(1980), 1 -9.

[4] P.A.Fillmore, Notes on Operator Theory, D. Van Nostrand, New York, 1970.

[5] P. R. Halmos, A Hilbert Space Problem Book, Van Nostrand, New York,1999.

[6] P.R. Halmos, A Hilbert Space Problem Book, D. Van Nostrand, New York, 1967.

[7] A. Lebow, On Von Neumann's Theory of Spectral Sets, J. Math. Anal. Appl. 7(1963), 64-90.