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A maximum principle for minimal hypersurfaces[1]

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Abstract

We study a geometric version of the maximum principle for minimal hypersurfaces.

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1. Introduction

The maximum principle is one of the most used tools in the study of some differential equations of elliptic type. It is a generalization of the following well known theorem of the elemental calculus "If f is a function of class C^2 in [a,b] such that the second derivative is positive on (a,b), then the maximum value of f attains at the ends of [a,b]". It is important to point out that the maximum principle gives information about the global behavior of a function over a domain from the information of qualitative character on the boundary and without explicit knowledge of the same function. The maximum principle allows us, for example, to obtain uniqueness of solution of certain problems with conditions of the Dirichlet and Neumann type. Also it allows to obtain a priori estimates for solutions. These reasons make interesting the study of the maximum principle on several forms and its generalizations. For example a geometric version of the maximum principle allows us to compare surfaces locally that coincide at a point. On the other hand, the maximum principle and the Alexandrov reflection principle have been used to prove symmetries with respect to some point, some plane, symmetries of domain and to determine asymptotic–symmetric behavior of the solutions of some elliptic problems. (See Berestycki and Nirenberg [1], Caffarelli, Gidas and Spruck [2], Gidas, Ni and Nirenberg [4], [5], Serrin [9]). J. Serrin was the first person who use this technic. Serrin proved that: "If u is a positive solution of the problem

$$\begin{aligned} \Delta u &= -1 \quad \text{on} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \\ \partial u/\partial\eta &= \text{Constant} \quad \text{on} \quad \partial\Omega, \end{aligned}$$

then Ω is a ball and u is radially symmetric with respect to the center of Ω" Using the ideas of Serrin and a version of the maximum principle for functions that do not change of sign suggested by himself Serrin, Gidas Ni and Nirenberg proved that: "If Ω is a ball, $f \in C^1(\mathbb{R})$ and u is a positive solution of the problem,

$$\begin{aligned} \Delta u + f(u) &= 0 \quad \text{on} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

then u is radially symmetric with respect to the center of the ball" Using the method of reflection and a version of maximum principle for thin domains Beresticky and Nirenberg generalized the paper [4]. In this paper we prove a geometric version of the maximum principle for minimal hypersurfaces.

2. Preliminaries and maximum principle

Below we show the maximum principle for the operator $L[u] = \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i}$ where $x = (x_1, \cdots, x_n)$ is in a domain $\Omega \subseteq \mathbb{R}^n, n \geq 2$. Suppose also that the coefficients $a_{ij}(x), b_i(x)$ are bounded in Ω, and $a_{ij}(x) = a_{ji}(x)$, for $i, j = 1, 2, \cdots, n$.

Theorem 2.1. Let $u \in C^2(\Omega)$ be such that $L[u] > 0$. If L is elliptic in Ω, then u does not attain its maximum value in Ω.

For the last we study the geometric version of the maximum principle for hypersurfaces.

3. Hypersurfaces in $\mathbb{R}^{n+1}$

Definition 3.1. A subset $\mathfrak{O}$ of $\mathbb{R}^{n+1}$ is called a hypersurface C^k of $\mathbb{R}^{n+1}$ if locally $\mathfrak{O}$ can be represented as the graphic of a function on an open subset of $\mathbb{R}^n$.

Namely, for each $p \in \mathfrak{O}$ there exists a neighborhood U of p in $\mathbb{R}^{n+1}$, a reordering of variables $x_0, \cdots, x_n$, an open set V of $\mathbb{R}^n$, and a function $h \in C^2(V)$ such that $\mathfrak{O} \cap U = \{(x', h(x')) : x' \in V\}$, where $x' = (x_0, \cdots, x_{n-1})$.

We need the following notation: Let $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection map $p(x, x_{n+1}) = x$, and for $t \in \mathbb{R}$ π_t is the hyperplane $\{x_{n+1} = t\}$ such that $\mathbb{R}^n \approx \pi_0$. For any set $S \subset \mathbb{R}^{n+1}$, we say that S is a graphic if the projection of S in $\mathbb{R}^{n+1}$ is one to one. Finally, if A, $B \subseteq \mathbb{R}^{n+1}$ we say that $A \geq B$ if for each $x \in \mathbb{R}^n$ for which $p^{-1}\{x\} \cap A \neq 0$ and $p^{-1}\{x\} \cap B \neq 0$, for all points of $p^{-1}\{x\} \cap A$ are over all points of $p^{-1}\{x\} \cap B$. Namely, if $(x, x_{n+1}) \in p^{-1}\{x\} \cap A$ and $(x, y_{n+1}) \in p^{-1}\{x\} \cap B$, then $x_{n+1} \geq y_{n+1}$.

3.1. Mean curvature of a hypersurface of class C^2

Definition 3.2. If $\mathfrak{O}$ is the graphic in $\mathbb{R}^{n+1}$ of a function of n variables $u \in C^2(\Omega)$, namely, $\mathfrak{O}$ is define by $x_{n+1} = u(x_1, \cdots, x_n)$, the mean curvature of $\mathfrak{O}$ at $x_0 \in \Omega$ is given by

$$H(x_0) = \frac{1}{n}\left[\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right)\right]_{x=x_0}.$$

$\mathfrak{O}$ is called a minimal hypersurface if $H(x_0) = 0$.

4. Main result

It is possible to compare hypersurfaces of class C^2 that coincide on the boundary using the maximum principle, if it knows the mean curvature in each of them. The following assertion is true.

Theorem 4.1. Suppose that 0 is an interior point of M_1 and M_2, and also $T_0M_1 = T_0M_2 = \{x_{n+1} = 0\}$. If $H_1 = 0$ and $H_2 = 0$ close to 0, then it is not true that $M_1 \geq M_2$ close to 0 at least that $M_1 = M_2$ in a neighborhood of 0.

Proof. We observe that if M_1 and M_2 are graphics of f and g respectively, then from hypothesis $H_1 = 0$, we have

$$\frac{1}{n}\left[\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{f_i}{\sqrt{1+|\nabla f|^2}}\right)\right]=0,$$

namely,

$$\frac{\sum_{i=1}^{n} f_{ii}(1+|\nabla f|^2)^{\frac{1}{2}} - \sum_{i=1}^{n}(f_i(1+|\nabla f|^2)^{\frac{1}{2}}).\sum_{j=1}^{n} f_j f_{ij}}{1+|\nabla f|^2}=0.$$

Therefore

$$\frac{(1+|\nabla f|^2)^{\frac{1}{2}}.\sum_{i,j=1}^{n}\left(f_{ii}-\frac{f_i f_j f_{ij}}{1+|\nabla f|^2}\right)}{1+|\nabla f|^2}=0.$$

In consequence

$$\sum_{i,j=1}^{n}\left(\delta_{ij}-\frac{f_i f_j}{1+|\nabla f|^2}\right).f_{ij}=0.$$

Writing $L_1 = \sum a_{ij}^1(x) f_{ij} = 0$ where

$$a_{ii}^1(x) = 1-\frac{f_i f_j}{1+|\nabla f|^2}$$

$$a_{ij}^1(x) = a_{ji}^1(x) = -\frac{f_i f_j}{1+|\nabla f|^2}$$

As (a_{ij}^1) is symmetric and it can prove that it is defined positive whatever be the function f, the differential equation associated with the operator L_1 is elliptic. In the same manner, as $H_2 = 0$, we have

$$\sum_{i,j=1}^{n}\left(\delta_{ij}-\frac{g_i g_j}{1+|\nabla f|^2}\right).g_{ij}=0.$$

Now, taking $u = f - g$ we have

$$\sum_{i,j=1}^{n}\left\{\left(\delta_{ij}-\frac{f_i f_j}{1+|\nabla f|^2}\right).(f_{ij}-g_{ij})-\left(\frac{f_i f_j}{1+|\nabla f|^2}-\frac{g_i g_j}{1+|\nabla g|^2}\right).g_{ij}\right\}=0.$$

Then

$$\sum_{i,j=1}^{n} a_{ij}u_{ij} - \sum_{j,k=1}^{n}(\beta_{jk}(f_1,\cdots,f_n)-\beta_{jk}(g_1,\cdots,g_n))g_{jk}=0,$$

where $\beta_{jk}(p_1, \cdots, p_n) = \dfrac{p_j p_k}{1+|p|^2}, a_{ij} = \delta_{ij} - \dfrac{f_i f_j}{1+|\nabla f|^2}$. By the mean value theorem,

$$\sum_{i,j=1}^{n} a_{ij}u_{ij} - \sum_{j,k=1}^{n} \left(\int_0^1 \frac{d}{dt}\beta_{jk}(\nabla g + t(\nabla f - \nabla g))dt \right) g_{jk} = 0.$$

Hence

$$\sum_{i,j=1}^{n} a_{ij}u_{ij} - \sum_{j,k=1}^{n} \left(\int_0^1 \mathrm{grad}(\beta_{jk}(\nabla g + t(\nabla f - \nabla g)))(\nabla f - \nabla g)dt \right) g_{jk} = 0.$$

In consequence

$$\sum_{i,j=1}^{n} a_{ij}u_{ij} - \sum_{i=1}^{n} \sum_{j,k=1}^{n} \left(\int_0^1 \frac{\partial \beta_{jk}}{\partial p_i}(\nabla g + t(\nabla f - \nabla g))dt . g_{jk} \right) u_i = 0.$$

Then

$$L[u] \equiv \sum_{i,j=1}^{n} a_{ij}u_{ij} + \sum_{i=1}^{n} b_i u_i = 0,$$

where

$$a_{ji} = a_{ij} = \delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2}, \quad b_i = -\sum_{j,k=1}^{n} \left(\int_0^1 \frac{\partial \beta_{jk}}{\partial p_i}(\nabla g + t(\nabla f - \nabla g))dt \right) . g_{jk},$$

$$a_{ii} = 1 - \frac{(f_i)^2}{1+|\nabla f|^2}.$$

As a_{ij} is symmetric and it can prove that it is positive defined whatever the function f, the linear differential equation associated with the operator L is elliptic. As f, g are C^2 and $L[u]$ is elliptic and $u(x) = f(x) - g(x) \geq u(0) = f(0) - g(0) = 0$ for all $x \in V$ and $0 \in int(V)$, the minimum of u attains in V, by the maximum principle we have that if M_1, M_2 are minimal hypersurfaces with boundaries B_1 and B_2, and if there exists a point 0 interior to B_1 and B_2, then it is possible to conclude that M_1 and M_2 coincide in a neighborhood of 0, namely, $f(x) = g(x)$ for $x \in V$. ■

References

[1] H. Berestycki and L. Nirenberg, On the method of moving planes and the sliding method, *Bol. Soc. Brasil. Mat.*, 22(1) (1991), 1–37.

[2] L. Cafarelli, B. Gidas and J. Spruck, Asymptotic symetry and local behavior of semilinear elliptic with critical sobolev growth, *Comm Math. Phys.*, 42 (1989), 271–297.

[3] M. Do Carmo, *Riemannian geometry*, Birkhäuser (1992).

[4] B. Gidas, Wei-Ming Ni and L. Nirenberg, Symetry and related properties via maximun principle, *Comm Math. Phys.*, 68 (1979), 209–243.

[5] B. Gidas, Wei-Ming Ni and L. Nirenberg, Symetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, *in Mathematical Analysis and Applications, Part A, in: Adv. Math. Suppl. Stud. vol 7A*, Academic Press, New York, (1981), 369–402.

[6] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, Berlin, Heidelberg, New York (1977).

[7] F. John, *Partial Differential Equations*, Springer Verlag, New York (1982).

[8] M. Protter and H. Weinberger, *Maximun Principle in Differential Equations*, Springer–Verlag, New York (1984).

[9] J. Serrin, A Symetry problem in potential theory, *Arch. Ration. Mech. Anal*, **43** (1971), 304–318.