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Neighborhood Properties of Analytic Functions Involving Multiplier Transformation

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Abstract

In this note, the subclasses $\mathcal{S}_m(\beta,\gamma,\lambda,l), \mathcal{R}_m(\beta,\gamma,\lambda,l;\mu)$, $\mathcal{S}_m^\alpha(\beta,\gamma,\lambda,l)$ and $\mathcal{R}_m^\alpha(\beta,\gamma,\lambda;\mu)$ of $\mathcal{A}(n)$ are defined and some properties of neighborhoods are studied for functions of complex order in these classes.

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1. Introduction

Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0, k \in \mathbb{N}\backslash\{1\}, n \in \mathbb{N}). \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. For any function $f \in \mathcal{A}(n)$ and $\delta \geq 0$, we define,

$$\mathcal{N}_{n,\delta}(f) = \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{and} \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \right\} \tag{2}$$

which is the (n, δ) - neighborhood of $f(z)$.

For $e(z) = z$, we see that,

$$\mathcal{N}_{n,\delta}(e) = \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{and} \sum_{k=n+1}^{\infty} k|b_k| \leq \delta \right\}. \tag{3}$$

The concept of neighborhoods was first introduced by Goodman [4] and then generalized by Ruscheweyh [6].

Let $\mathcal{S}_n^*(\gamma)$[1] denote the subclass of $\mathcal{A}(n)$, defined by the functions of complex order γ satisfying,

$$\Re\left\{1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} > 0, \quad (z \in \mathcal{U}, \gamma \in \mathbb{C}\backslash\{0\}). \tag{4}$$

The subclass $\mathcal{C}_n(\gamma)$[1] of $\mathcal{A}(n)$, is the class of functions of complex order γ satisfying,

$$\Re\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathcal{U}, \gamma \in \mathbb{C}\backslash\{0\}). \tag{5}$$

For $f \in \mathcal{A}(n)$, multiplier transformation $I(m, \lambda, l)f(z)$ [3] defined by

$$I(m, \lambda, l)f(z) = z - \sum_{k=n+1}^{\infty} \left(\frac{\lambda(k-1) + l + 1}{l+1}\right)^m a_k z^k, m \in N_0, \lambda, l \geq 0, n \in \mathbb{N}. \tag{6}$$

For $l = 0$, multiplier transformation reduces to Al-Oboudi differential operator [2]. Now using multiplier transformation, we define the following subclasses of $\mathcal{A}(n)$. The subclass $\mathcal{S}_m(\beta, \gamma, \lambda, l)$ of $\mathcal{A}(n)$ is the class of functions f such that

$$\left|\frac{1}{\gamma}\left(\frac{z(I(m, \lambda, l)f(z))'}{I(m, \lambda, l)f(z)} - 1\right)\right| < \beta, \tag{7}$$

where $m \in \mathbf{N}_0, \quad \gamma \in \mathbb{C}\backslash\{0\}, \quad 0 < \beta \leq 1, \quad \lambda, l \geq 0$ and $z \in \mathcal{U}$.

And let the subclass $\mathcal{R}_m(\beta, \gamma, \lambda, l; \mu)$ of $\mathcal{A}(n)$ be the class of functions f such that

$$\left|\frac{1}{\gamma}\left((1-\mu)\frac{I(m, \lambda, l)f(z)}{z} + \mu(I(m, \lambda, l)f(z))' - 1\right)\right| < \beta, \tag{8}$$

where, $m \in \mathbf{N}_0, \quad \gamma \in \mathbb{C}\backslash\{0\}, \quad 0 < \beta \leq 1 \quad \lambda, l \geq 0$ and $z \in \mathcal{U}$.

In the following sections, we obtain inclusion relations involving (n, δ) for analytic functions in the classes $S_m(\beta, \gamma, \lambda, l)$ and $R_m(\beta, \gamma, \lambda, l; \mu)$.

2. Main Results

Lemma 2.1. A function $f \in \mathcal{S}_m(\beta, \gamma, \lambda, l)$ if and only if

$$\sum_{k=n+1}^{\infty} \left(\frac{\lambda(k-1) + l + 1}{l+1}\right)^m [\beta|\gamma| + k - 1]a_k < \beta|\gamma|. \tag{9}$$

Proof. Suppose $f \in \mathcal{S}_{m,n}(\beta, \gamma, \lambda)$. Then by (7) we can write

$$\Re\left\{\frac{z(I(m,\lambda,l)f(z))'}{I(m,\lambda,l)f(z)} - 1\right\} > -\beta|\gamma|, \quad (z \in \mathcal{U}). \tag{10}$$

Using (1) and (6), we have,

$$\Re\left\{\frac{\sum_{k=n+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k z^k [1-k]}{z - \sum_{k=n+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k z^k}\right\} > -\beta|\gamma|.$$

Let $z \to 1$, through the real values in the above inequality we get

$$\sum_{k=n+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k(1-k) > -\beta|\gamma|\left(1 - \sum_{k=n+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k\right).$$

This inequality yields the desired condition. Conversely, by (9) and letting$|z| = 1$ we obtain,

$$\begin{aligned}
\left|\frac{z(I(m,\lambda,l)f(z))'}{I(m,\lambda,l)f(z)} - 1\right| &= \left|\frac{\sum_{k=n+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k z^k(1-k)}{z - \sum_{k=n+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k z^k}\right| \\
&\le \frac{\beta|\gamma|\left(1 - \sum_{k=n+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k(k-1)\right)}{1 - \sum_{k=n+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k} \\
&\le \beta|\gamma|.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f \in \mathcal{S}_m(\beta, \gamma, \lambda, l)$, which established the required result. ■

On similar lines, we have the following Lemma.

Lemma 2.2. A function $f \in \mathcal{R}_m(\beta, \gamma, \lambda, l; \mu)$ if and only if

$$\sum_{k=n+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [\mu(k-1)+1]a_k < \beta|\gamma|. \tag{11}$$

Theorem 2.3. If $\delta = \dfrac{\beta|\gamma|(n+1)}{\left(\frac{(n\lambda+l+1)}{l+1}\right)^m [\beta|\gamma|+n]}$, $(|\gamma| < 1)$, then $\mathcal{S}_m(\beta,\gamma,\lambda,l) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. Let $f \in \mathcal{S}_m(\beta,\gamma,\lambda,l)$. Then, by Lemma 2.1, we have

$$\sum_{k=n+1}^{\infty} a_k \le \frac{\beta|\gamma|}{[\beta|\gamma|+n]\left(\frac{\lambda n+l+1}{l+1}\right)^m}. \tag{12}$$

Using (9) and (12), we have,

$$\begin{aligned}\left(\frac{(n\lambda+l+1)}{l+1}\right)^m \sum_{k=n+1}^{\infty} ka_k &\le \beta|\gamma| + (1-\beta|\gamma|)\left(\frac{(n\lambda+l+1)}{l+1}\right)^m \sum_{k=n+1}^{\infty} a_k \\ &\le \frac{\beta|\gamma|(n+1)}{[\beta|\gamma|+n]}.\end{aligned}$$

$$i.e., \sum_{k=n+1}^{\infty} ka_k \le \frac{\beta|\gamma|(n+1)}{[\beta|\gamma|+n]\left(\dfrac{(n\lambda+l+1)}{l+1}\right)^m} = \delta.$$

Thus, by the definition given by (3), $f \in \mathcal{N}_{n,\delta}(e)$. Hence the proof. ∎

Theorem 2.4. If $\delta = \dfrac{\beta|\gamma|(n+1)}{[\mu n+1]\left(\frac{(n\lambda+l+1)}{l+1}\right)^m}$, $|\gamma| < 1$, then $\mathcal{R}_m(\beta,\gamma,\lambda,l;\mu) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. Let $f \in \mathcal{R}_m(\beta,\gamma,\lambda,l;\mu)$. By Lemma 2.2, we have,

$$\sum_{k=n+1}^{\infty} a_k \le \frac{\beta|\gamma|}{[\mu n+1]\left(\frac{(n\lambda+l+1)}{l+1}\right)^m}. \tag{13}$$

From (11) and (13), we have,

$$\begin{aligned}\mu\left(\frac{(n\lambda+l+1)}{l+1}\right)^m \sum_{k=n+1}^{\infty} ka_k &\le \beta|\gamma| + (\mu-1)\left(\frac{(n\lambda+l+1)}{l+1}\right)^m \sum_{k=n+1}^{\infty} a_k \\ &\le \frac{\beta|\gamma|\mu(n+1)}{[\mu n+1]}.\end{aligned}$$

That is,

$$\sum_{k=n+1}^{\infty} ka_k \le \frac{\beta|\gamma|(n+1)}{[\mu n+1]\left(\dfrac{(n\lambda+l+1)}{l+1}\right)^m} = \delta.$$

From (3), we have, $f \in \mathcal{N}_{n,\delta}(e)$, which completes the proof. ∎

3. Neighborhoods for classes $\mathcal{S}_m^\alpha(\beta,\gamma,\lambda,l)$ and $\mathcal{R}_m^\alpha(\beta,\gamma,\lambda,l;\mu)$

In this section, we define the subclasses $\mathcal{S}_m^\alpha(\beta,\gamma,\lambda,l)and\mathcal{R}_m^\alpha(\beta,\gamma,\lambda,l;\mu)of\mathcal{A}(n)$ and neighborhoods of these classes are obtained.

For $0 \leq \alpha < 1$ and $z \in \mathcal{U}$, a function $f \in \mathcal{S}_m^\alpha(\beta,\gamma,\lambda,l)$ if there exists a function $g(z) \in \mathcal{S}_m(\beta,\gamma,\lambda,l)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \alpha. \tag{14}$$

For $0 \leq \alpha < 1$ and $z \in \mathcal{U}$, a function $f \in \mathcal{R}_m^\alpha(\beta,\gamma,\lambda,l;\mu)$ if there exists a function $g(z) \in \mathcal{R}_m(\beta,\gamma,\lambda,l;\mu)$ such that the inequality (14) holds true.

Theorem 3.1. If $g(z) \in \mathcal{S}_m(\beta,\gamma,\lambda,l)$ and

$$\alpha = 1 - \frac{\delta[\beta|\gamma| + n]\left(\frac{(n\lambda + l + 1)}{l+1}\right)^m}{(n+1)\left[(\beta|\gamma| + n)\left(\frac{(n\lambda + l + 1)}{l+1}\right)^m - \beta|\gamma|\right]}, \tag{15}$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{S}_m^\alpha(\beta,\gamma,\lambda,l)$.

Proof. Let $f \in \mathcal{N}_{n,\delta}(g)$. Then,

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta, \tag{16}$$

which yields the coefficient inequality,

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1}, \quad (n \in \mathbb{N}). \tag{17}$$

Since $g(z) \in \mathcal{S}_m(\beta,\gamma,\lambda,l)$ by (12), we have,

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\beta|\gamma|}{[\beta|\gamma| + n]\left(\frac{(n\lambda + l + 1)}{l+1}\right)^m}, \tag{18}$$

so that,

$$\begin{aligned}\left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k} \\ &\leq \frac{\delta[\beta|\gamma| + n]\left(\frac{(n\lambda + l + 1)}{l+1}\right)^m}{(n+1)\left[(\beta|\gamma| + n)(n\lambda + 1)^m - \beta|\gamma|\right]} \\ &= 1 - \alpha.\end{aligned}$$

Thus by definition, $f \in \mathcal{S}_m^{\alpha}(\beta, \gamma, \lambda, l)$ for a given α by (15), which establishes the desired result. ∎

On similar lines, we can prove the following Theorem.

Theorem 3.2. If $g(z) \in \mathcal{R}_m^{\alpha}(\beta, \gamma, \lambda, l; \mu)$ and

$$\alpha = 1 - \frac{\delta[\mu n + 1]\left(\dfrac{(n\lambda + l + 1)}{l+1}\right)^m}{(n+1)\left[[\mu n + 1]\left(\dfrac{(n\lambda + l + 1)}{l+1}\right)^m - \beta|\gamma|\right]}, \tag{19}$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{R}_m^{\alpha}(\beta, \gamma, \lambda, l; \mu)$.

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