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Catalan-type Fermat Numbers

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Abstract

In this paper we consider Catalan-type Fermat primes as another case to suit with our previous Conjecture 2 (Fermat primes or special Fermat primes are infinite if both the sum of corresponding original continuous natural number sequence and the first such prime are Mersenne primes, but such primes are absolutely finite if one of them is not Mersenne prime)[1], and get corresponding corollary which implies Catalan-type Fermat primes to be absolutely finite, that is, there exist only four Catalan-type Fermat primes 3,5,17,65537. We discover existence of the congruences to be suitable to Catalan-type Fermat numbers to be a special case of the congruences to be suitable to double Fermat numbers, present the congruences to be suitable to double Fermat numbers are a new and equivalent statement of Fermat's little theorem for Fermat numbers, and expect there may be infinitely many strong pseudoprimes to base 2.

Keywords: Fermat number; double Fermat number; Catalan-type Fermat number; strong pseudoprime to base 2; Fermat's little theorem

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1. Previous Main Results

In our previous work[1] we got two conjectures to indicate existence of mutual restricting connections between Mersenne and Fermat primes. Conjecture 1 stated that Mersenne primes or special Mersenne primes are infinite if both the sum of corresponding original continuous prime number sequence and the first such prime are Fermat primes, but such primes are absolutely finite if one of them is not Fermat prime. Conjecture 2 stated that Fermat primes or special Fermat primes are infinite if both the sum of corresponding original continuous natural number sequence and the first such prime are Mersenne primes, but such primes are absolutely finite if one of them is not Mersenne prime. So-called absolute finiteness of such a kind of primes means that there exist no any other such primes besides original continuous prime or natural number sequence in corresponding basic sequence of number. By Conjecture 1 we got three corollaries which imply Mersenne primes and root Mersenne primes to be infinite but double Mersenne primes to be absolutely finite, and the last result means there exist no any double Mersenne numbers $2^{M_p}-1$ to be prime for $p > 7$. By Conjecture 2 we got two corollaries which imply Fermat and double Fermat primes to be all absolutely finite, that is, Fermat numbers F_n for n = 0,1,2,3,4 i.e. 3,5,17,257,65537 are all prime but every Fermat number is composite for $n > 4$, and double Fermat numbers $F_{2^n} = 2^{F_n-1}+1$ for n = 0,1,2 i.e. 5,17,65537 are all prime but every double Fermat number is composite for $n > 2$. We discovered existence of the recurrence relations $F_{2^{n+1}} = (F_{2^n}-1)^{F_n-1}+1$ with $F_{2^0} = 5$ for $n \geq 0$ to be suitable to double Fermat numbers.

2. Absolute Finiteness of Catalan-type Fermat Primes

When noticing mathematical form of Catalan-Mersenne numbers[2], we can discover existence of a similar form in Fermat numbers, that is, there exists a kind of special Fermat numbers to satisfy the recurrence relations $Fc\,(n+1) = 2^{Fc(n)-1}+1$ with $Fc\,(0) = F_0 = 3$ for $n \geq 0$, where n is natural number and $Fc\,(n)$ can be called Catalan-type Fermat numbers. An anonymous writer proposed that such numbers were all prime, however, this conjecture was refuted when Selfridge showed F_{16} is composite in 1953[3]. Catalan-type Fermat numbers are infinite and grow very quickly:

$Fc\,(0) = F_0 = 3$

$Fc\,(1) = 2^{Fc(0)-1}+1 = F_{2^0} = F_1 = 5$

$Fc\,(2) = 2^{Fc(1)-1}+1 = F_{2^1} = F_2 = 17$

$Fc\,(3) = 2^{Fc(2)-1}+1 = F_{2^2} = F_4 = 65537$

$Fc\,(4) = 2^{Fc(3)-1}+1 = F_{2^4} = F_{16} = 2^{65536}+1$

$Fc\,(5) = 2^{Fc(4)-1}+1 = F_{2^{16}} = F_{65536}$

$Fc\,(6) = 2^{Fc(5)-1}+1 = F_{2^{65536}}$

Catalan-type Fermat numbers are a subsequence of Fermat numbers but are not a subsequence of double Fermat numbers since the first Catalan-type Fermat number $Fc\,(0) = F_0$ is a Fermat number but is not a double Fermat number though $Fc\,(1)$ and all of the following Catalan-type Fermat numbers are double Fermat numbers. We have seen Fermat, double Fermat and Catalan-type Fermat numbers all satisfy corresponding recurrence relations, that is, $F_{n+1} = (F_n-1)^2+1$ with $F_0 = 3$ for $n \geq 0$ are suitable to Fermat numbers[4], $F_{2^{n+1}} = (F_{2^n}-1)^{F_n-1}+1$ with $F_{2^0} = 5$ for $n \geq 0$ are suitable to double Fermat numbers and $Fc\,(n+1) = 2^{Fc(n)-1}+1$ with $Fc\,(0) = 3$ for $n \geq 0$ are suitable to Catalan-type Fermat numbers.

We have known Catalan-type Fermat numbers can be thought a kind of special

Fermat numbers as double Fermat numbers do, and we have seen the first four Catalan-type Fermat numbers $Fc\,(n)$ for n = 0,1,2,3 i.e. 3,5,17,65537 are all prime but the fifth Catalan-type Fermat number $Fc\,(4) = F_{16}$ is known composite Fermat number to indicate $Fc\,(4)$ to be composite Catalan-type Fermat number. Catalan-type Fermat primes are a kind of special Fermat primes and arise from Catalan-type Fermat numbers $Fc\,(n)$ so that all of natural numbers can be called basic sequence of number[1] of Catalan-type Fermat primes, which is an infinite sequence of number. If the first few continuous natural numbers make $Fc\,(n)$ become Catalan-type Fermat primes then these natural numbers can be called original continuous natural number sequence[1] of Catalan-type Fermat primes. Considering $Fc\,(n)$ for n = 0,1,2,3 to be Catalan-type Fermat primes but $Fc\,(4)$ not to be Catalan-type Fermat prime, we can confirm there exists an original continuous natural number sequence of Catalan-type Fermat primes i.e. n = 0,1,2,3. Hence we get following corollary.

Corollary 3 (Conjecture 2[1]). If Conjecture 2[1] is true, then Catalan-type Fermat primes are absolutely finite.

Proof. Since the first Catalan-type Fermat prime $Fc\,(0)$ =3 is a Mersenne prime i.e. M_2 but the sum of original continuous natural number sequence of Catalan-type Fermat primes i.e. 0+1+2+3 = 6 is not a Mersenne prime, we will get the result.

Corollary 3 (Conjecture 2[1]) implies Catalan-type Fermat primes are absolutely finite, that is, there may exist no any other Catalan-type Fermat primes besides n = 0,1,2,3 in basic sequence of number of Catalan-type Fermat primes though the basic sequence of number is infinite. It means every Catalan-type Fermat number is composite for $n > 3$.

The Congruences to Be Suitable to Catalan-type Fermat Numbers

It is well known that every composite Fermat number is a strong pseudoprime to base 2 because all strong pseudoprimes to base 2 are also Fermat pseudoprimes i.e. $2^{F_n-1} \equiv 1 \pmod{F_n}$ for all Fermat numbers[4], which are statement of Fermat's little theorem for Fermat numbers. Thus by Corollary 1 (Conjecture 2)[1] to imply every Fermat number to be composite for $n > 4$ we may expect that there are infinitely many strong pseudoprimes to base 2 to be generated from the Fermat numbers, in other words, if Corollary 1 (Conjecture 2)[1] holds then there are infinitely many strong pseudoprimes to base 2. Further, by $2^{F_n-1} \equiv 1 \pmod{F_n}$ and $F_{2^n} = 2^{F_n-1}+1$ we get the congruences $F_{2^n} \equiv 2 \pmod{F_n}$ to be suitable to double Fermat numbers F_{2^n} generated from Fermat numbers F_n such as $F_{2^0} \equiv 2 \pmod{F_0}$, $F_{2^1} \equiv 2 \pmod{F_1}$, $F_{2^2} \equiv 2 \pmod{F_2}$, $F_{2^3} \equiv 2 \pmod{F_3}$, $F_{2^4} \equiv 2 \pmod{F_4}$ and $F_{2^5} \equiv 2 \pmod{F_5}$. The congruences $F_{2^n} \equiv 2 \pmod{F_n}$ are a new and equivalent statement of Fermat's little theorem for Fermat numbers by introducing double Fermat number formula $F_{2^n} = 2^{F_n-1}+1$ and present the property involving Fermat numbers that F_m-2 is divisible by all smaller Fermat numbers i.e. $F_m = F_0\, F_1\, F_2 \ldots F_{m-1} + 2$[5] but $m = 2^n$ here.

Considering $Fc\,(n+1)$ to be always a double Fermat number generated from $Fc\,(n)$ to be a Fermat number, we get the congruences $Fc\,(n+1) \equiv 2 \pmod{Fc\,(n)}$ to be suitable to Catalan-type Fermat numbers. From it we get following infinite sequence to indicate Catalan-type Fermat numbers to satisfy such congruences.

$Fc(1) \equiv 2$ (mod $Fc(0)$) i.e. $F_{2^0} \equiv 2$ (mod F_0)

$Fc(2) \equiv 2$ (mod $Fc(1)$) i.e. $F_{2^1} \equiv 2$ (mod F_1)

$Fc(3) \equiv 2$ (mod $Fc(2)$) i.e. $F_{2^2} \equiv 2$ (mod F_2)

$Fc(4) \equiv 2$ (mod $Fc(3)$) i.e. $F_{2^4} \equiv 2$ (mod F_4)

$Fc(5) \equiv 2$ (mod $Fc(4)$) i.e. $F_{2^{16}} \equiv 2$ (mod F_{16})

$Fc(6) \equiv 2$ (mod $Fc(5)$) i.e. $F_{2^{65536}} \equiv 2$ (mod F_{65536})

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Conclusion

In this paper by Conjecture 2[1] we showed Catalan-type Fermat primes are absolutely finite i.e. there are only four Catalan-type Fermat primes 3,5,17,65537. It means every Catalan-type Fermat number is composite for $n > 3$. We found the congruences $Fc(n+1) \equiv 2$ (mod $Fc(n)$) to be suitable to Catalan-type Fermat numbers, which are a special case of the congruences $F_{2^n} \equiv 2$ (mod F_n) to be suitable to double Fermat numbers and also a special case of the property that $F_m - 2$ is divisible by all smaller Fermat numbers. We expected existence of infinitely many strong pseudoprimes to base 2 by Corollary 1 (Conjecture 2)[1], and pointed out that the congruences $F_{2^n} \equiv 2$ (mod F_n) can be thought a new and equivalent statement of Fermat's little theorem for Fermat numbers by using double Fermat number formula $F_{2^n} = 2^{F_n - 1} + 1$.

References

[1] Pingyuan Zhou, On the Connections between Mersenne and Fermat Primes, Global Journal of Pure and Applied Mathematics, Vol.8, No.4(2012), 453-458.
[2] Catalan-Mersenne Number in The On-Line Wolfram MathWorld.
[3] Fermat Number in The On-Line Wolfram MathWorld.
[4] Fermat number in The On-Line Wikipedia.
[5] Fermat numbers in The On-Line PlanetMath.