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Integral solutions of the quadratic with four unknowns (x + y)(z + w) = xy + 4zw

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Abstract

The quadratic equation with four unknowns (x + y)(z + w) = xy + 4zw is analysed for non trivial integral solutions. A few interesting relations between the solutions and the special numbers are presented.

Keywords Integral solutions , Quadratic with four unknowns.

MSC 2000 Subject Classification number : 11D09

Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous or non-homogeneous quadratic Diophantine equations with two or more variables have been an interest to mathematicians since antiquity [1-8]. In this context, one may refer [9-17] for different choices of quadratic diophantine equations with four unknowns. This communication concerns with yet another interesting parametric integral solutions of the quadratic equation with four unknowns (x + y)(z + w) = xy + 4zw is analyzed for different patterns of non-zero distinct integer solutions. Given a solution, a general formula for generating a sequence of integer solutions is also exhibited.

Polygonal Numbers	Notations for rank 'n'	Definitions
Triangular number of rank n	T_n	$\frac{1}{2}n(n+1)$
Pentagonal number of rank n	Pen_n	$\frac{1}{2}(3n^2-n)$
Hexagonal number of rank n	Hex_n	$2n^2-n$
Heptagonal number	Hep_n	$\frac{1}{2}(5n^2-3n)$
Octagonal number of rank n	Oct_n	$3n^2-2n$
Decagonal number of rank n	Dec_n	$4n^2-3n$
Hendecagonal number	HD_n	$\frac{1}{2}(9n^2-7n)$
Dodecagonal number	DD_n	$\frac{1}{2}(10n^2-8n)$
Octadecagonal number	OD_n	$\frac{1}{2}(16n^2-14n)$
Icosagonal number	IC_n	$\frac{1}{2}(18n^2-16n)$
Gnomonic number of rank n	Gno_n	$2n-1$
Pronic number of rank n	Pro_n	$n(n+1)$
Stella Octangula number of rank n	SO_n	$n(2n^2-1)$
Star number of rank n	$Star_n$	$6n(n-1)+1$

Method of Analysis

The equation under consideration is

$$(x+y)(z+w) = xy + 4zw \quad (1)$$

To start with, it is noted that (1) is satisfied by the following quadraples: (3,2,1,1), (2t,t,t,t), (2t,s,t,t), (s,2t,t,t).

In addition to the above solutions, two more patterns of solutions are illustrated below:

Applying the transformations $x = u + p$, $y = u - p$, $z = p + q$ and $w = p - q$, equation (1) is reduced to

$$(u-2p)^2 = p^2 + (2q)^2 \quad \ldots\ldots \quad (2)$$

Pattern 1

Case 1: Let $2q = 2\alpha\beta,\ p = \alpha^2 - \beta^2\ \ and\ \ u - 2p = \alpha^2 + \beta^2$

Then the solutions of (1) are given by,

$$\left.\begin{aligned} x(\alpha,\beta) &= 4\alpha^2 - 2\beta^2 \\ y(\alpha,\beta) &= 2\alpha^2 \\ z(\alpha,\beta) &= \alpha^2 - \beta^2 + \alpha\beta \\ w(\alpha,\beta) &= \alpha^2 - \beta^2 - \alpha\beta \end{aligned}\right\} \quad (3)$$

Observations

1. 3y is a Nasty number.
2. 2(x - y) and 2(z + w) can be written as the difference of two perfect squares.
3. $x(\alpha,1) + 2$ is a perfect square.
4. x – y = z + w
5. $z(\alpha,1) - w(\alpha,1) = Gno_\alpha + 1$
6. $x(1,\beta) + z(1,\beta) + 2Pen_\beta \equiv 0 (\mathrm{mod}\ 5)$
7. $x(\alpha,1) - z(\alpha,1) + 2w(\alpha,1) = 2Hep_\alpha - 3$
8. $z(\alpha,1) - Pro_\alpha + 1 = 0$
9. $x(\alpha,1)y(\alpha,1) = 4Hex_{\alpha^2}$
10. $10z(\alpha,1)y(\alpha,1) + 20w(\alpha,1) - 10SO_\alpha - 4Oct_{\alpha^2} - 16T_{\alpha^2} + 5Gno_\alpha + 25 = 0$
11. $x(1,\beta) - w(1,\beta) + Oct_\beta - Hex_\beta \equiv 0 (\mathrm{mod}\ 3)$
12. $x(1,\beta)w(1,\beta) + Star_\beta - SO_\beta - Gno_{\beta^4} - 27Dec_\beta + 36Oct_\beta - 6 = 0$

Case 2: Let $2q = \alpha^2 - \beta^2,\ p = 2\alpha\beta,\ u - 2p = \alpha^2 + \beta^2$

Assume $\alpha = 2A, \beta = 2B$

Then $q = 2A^2 - 2B^2, p = 8AB\ \ and\ \ u = 4A^2 + 4B^2 + 16AB$

The solutions of (1) are given by,

$$\begin{aligned} x(A,B) &= 4A^2 + 4B^2 + 24AB \\ y(A,B) &= 4A^2 + 4B^2 + 8AB \\ z(A,B) &= 2A^2 - 2B^2 + 8AB \\ w(A,B) &= 2B^2 - 2A^2 + 8AB \end{aligned} \quad (4)$$

Observations

1. Each of the following is a Nasty number:

(i)6y is a Nasty number.
(ii)6(x - y)(z + w)

2. Each of the following represents a perfect square:
(i)2[x + y - z - w)] is a perfect square.
(ii) 2y(1,B) - 2w(1,B) - 12

3. $x(A,1)-8T_A-10Gno_A-14=0$

4. z - w can be written as the differnce of two perfect squares.

5. $y(A,1)-Dec_A-4\equiv 0 \pmod{11}$

6. $z(A,1)-4Gno_A-4Hex_A+2(Oct_A)-2=0$

7. $x(1,B)-w(1,B)-2\,Pro_B-7Gno_B\equiv 0 \pmod{13}$

Pattern 2

Let $u-2p=a^2+b^2$

Then, (2) gives $(a^2+b^2)^2 = p^2+(2q)^2$
which is written as

$$(a^2+b^2)^2 * 1 = p^2+(2q)^2 \tag{5}$$

Now write1 as $1 = \dfrac{(m^2-n^2+2mni)(m^2-n^2-2mni)}{(m^2+n^2)^2}$

Then (5) implies

$$(a^2+b^2)^2\frac{(m^2-n^2+2mni)(m^2-n^2-2mni)}{(m^2+n^2)^2} = p^2+(2q)^2 \tag{6}$$

Define $(a+ib)^2\dfrac{(m^2-n^2+2mni)}{(m^2+n^2)} = p+i(2q)$

By equating the real and imaginary parts on both sides we get,

$$p(m^2+n^2)=(m^2-n^2)(a^2-b^2)-4mnab$$

$$q(m^2+n^2)=(m^2-n^2)ab+mn(a^2-b^2)$$

For clear understanding, consider m = 2 , n = 1
Then the values of p and q are

$$p=\frac{1}{5}[3(a^2-b^2)-8ab]$$

$$q=\frac{1}{5}[3ab+2(a^2-b^2)]$$

Since our aim is to find integral solutions, let us choose a = 5A and b = 5B
Then

$$\left.\begin{aligned} p &= 15A^2 - 15B^2 - 40AB \\ q &= 10A^2 - 10B^2 + 15AB \\ u &= 55A^2 - 5B^2 - 80AB \end{aligned}\right\} \quad (7)$$

Therefore,the solutions x,y,z and w of (1) are given by

$$\left.\begin{aligned} x(A,B) &= 70A^2 - 20B^2 - 120AB \\ y(A,B) &= 40A^2 + 10B^2 - 40AB \\ z(A,B) &= 25A^2 - 25B^2 - 25AB \\ w(A,B) &= 5A^2 - 5B^2 - 55AB \end{aligned}\right\} \quad (8)$$

Observations

1. Each one of the following is a Nasty number:
   (i) $15y(A,1) + 300Gno_A + 150$
   (ii) $5[z(A,A) - w(A,A)]$
2. Each of the following is a perfect square:
   (i) $25Dec_A - 25Oct_A - z(A,1)$
   (ii) $z(1,B) + 50T_B$
3. $x(A,1) - 20Nan_A - 70Dec_A + 140Hex_A + 20 = 0$
4. $w(A,1) - DD_A + 5 \equiv 0 \pmod{51}$
5. $x(1,B) + 60Gno_B + 40HD_B - 20(OD_B) - 10 = 0$
6. $y(1,B) - 20DD_B + 10IC_B \equiv 0 \pmod{40}$
7. $w(1,B) + 5\Pr o_B + 25Gno_B + 20 = 0$

Conclusion

One may search for other patterns of solutions and the corresponding observations.

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