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Generalized Codes in Free Partial Monoids

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Abstract

In this paper we discuss the structure of certain kinds of strong semilattices of monoids, the so called free semilattices of monoids to obtain its properties analogous to the basic known properties of free monoids. We also define the notion of partial codes in free semilattices of monoids showing that every partial code is "biprefix" (i.e. prefix and suffix), and give characterizations of partial and maximal partial codes in an analogy to the known characterizations of codes and maximal codes in free monoids. This prepare the ground for studing generalized languages and autumata.

Key words and phrases. Free partial monoid, Partial code.

1 Introduction and Preliminaries

Perhaps one may recognize that the theory of languages and automata is based on certain properties of particular subsets of free monoids the so called rational sets. Besides the uniqueness of factorization of words in free monoids plays a central role in developing such a theory. It is thus not surprising that characterizations of submonoids of free monoids which are also free has obtained considerable interest from both semigroups and languages theorists. This naturally lead to study and characterizing certain types of codes, i.e. the bases or generating sets of free monoids. Prefix codes in particular has its significant role in the study of rational languages and finite automata . In nature, there are different languages expressed in terms of different (and disjoint) alphabets. Interaction between different languages may be viewed as mappings (or translators) between the corresponding alphabets. Formally, a natural model of the situation may have the structure of certain kinds of strong semilattices of monoids, the so called free semilattices of monoids. In the present work we discuss this structure to obtain its properties analogous to the basic known properties of free monoids. We also define the notion of partial codes in free semilattices of monoids showing that every partial code is "biprefix" (i.e. prefix and suffix), and give characterizations of partial and maximal partial codes in an analogy to the known characterizations of code and maximal codes in free monoids. In a subsequent paper we use the present work in developing particular kinds of generalized languages and autumata . For sake of reference and fixing notation we cite here some basic definitions and results needed for our work. A semilattice is a pair $(S, \leq)$ where $S$is a set and $\leq$ is a partial ordering on S (i.e. $\leq$ is a reflexive, antisymmytric and transitive relation on S), such that every pair of elements $a, b \in S$ has a greatest lower bound $a \wedge b$ in S. A semigroup S is called a band if every element $a \in S$ is idempotent i.e. $aa = a$ (or $a^2 = a$). We have : A semigroup S is a commutative band iff S (with the same partial ordering) is a semilattice. Actually if S is a commutative band, then the relation $\leq$ define on S by $a \leq b$ iff $ab = a$ turns S into a semilattice where for every pair $a, b \in S, a \wedge b$ is given by ab (the product of a, b in S). Conversely, if $(S, \leq)$ is a semilattice, then S is a commutative band with the operation in S defined by $\forall\ a, b \in S, ab = a \wedge b (= b \wedge a) = ba$. (Thus in particular $a^2 = a = a \wedge a = a$).

Let Y be a semilattice and let $\{S_\alpha : \alpha \in Y\}$be a collection of (disjoint) semigroups of particular type $\sqsubseteq$.

Then the disjoint union $S = \bigcup_{\alpha \in Y} S_\alpha$ is called a strong semilattice of semigroups S_α , $\alpha \in Y$ if for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$ there exists a homomorphism

$$\varphi_{\alpha,\beta} : S_\alpha \to S_\beta$$

such that $\varphi_{\alpha,\alpha}$ is the identical homomorphism, and for $\alpha \geq \beta \geq \gamma$ in Y,

$$\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma}.$$

We may write $S = [Y, S_\alpha, \varphi_{\alpha,\beta}]$ to indicate that S is a strong semilattice Y of semigroups $S_\alpha, \alpha \in Y$. If $S = [Y, S_\alpha, \varphi_{\alpha,\beta}]$, then there is a (unique) operation on S that extends the binary operation of S_α for every $\alpha \in Y$, given by, for $a, b \in S$, say $a \in S_\alpha, b \in S_\beta$

$$ab = \varphi_{\alpha,\alpha\beta} a \circ \varphi_{\beta,\alpha\beta} b.$$

Here the operation in RHS is the multiplication in $S_{\alpha\beta}$ of the elements $\varphi_{\alpha,\alpha\beta} a, \varphi_{\beta,\alpha\beta} b$ in $S_{\alpha\beta}$.

In particular a strong semilattice of groups is a Clifford semigroup (i.e. a regular semigroup with centeral idempotents). For our work we discuss the structure of strong semilattices of monoids. In the rest of this section we cite from [5] some requried materials.

A semigroup S is called an ε −semigroup (in words, epsilon semigroup) if it is equipped with a unary operation $\varepsilon: S \to S$, sending $x \mapsto \varepsilon_x$ with the following axioms satisfied: for all $x, y \in S$
(PM1) ε_x is idempotent (i.e. $\varepsilon_x\, \varepsilon_x = \varepsilon_x$)
(PM2) $\varepsilon_{(\varepsilon_x)} = \varepsilon_x$ (i.e. the operation ε is idempotent)
(PM3) $\varepsilon_x\, x = x\, \varepsilon_x = x$
(PM4) $\varepsilon_{(xy)} = \varepsilon_x \varepsilon_y$
the element ε_x is called the partial identity of x.

The subset of idempotents in an ε −semigroup S is denoted by $E(S)$, and the set of all partial identities in S , $\{\varepsilon_x : x \in S\}$ is denoted by $\varepsilon(S)$.

If S is an ε −semigroup, then by (PM1), $\varepsilon(S) \subset E(S)$ and so by (PM4), $\varepsilon(S)$ is idempotent subsemigroup of S.

A subset B of an ε −semigroup S is an ε −subsemigroup of S, if B is a subsemigroup of S and $\varepsilon_b \in B$ for every $b \in B$.

A mapping φ of an ε −semigroup S into an ε −semigroup T is an ε −homomorphism if it preserves the operations of S, i.e. $\varphi(xy) = \varphi(x)\varphi(y), x, y \in S$ and $\varepsilon_{(\varphi x)} = \varphi(\varepsilon_x)$, for all $x \in S$. Hence $\varepsilon_{(\varphi x)}$ is the partial identity in T of φx, and ε_x is the partial identity in S (of x). Clearly, the variety of ε −semigroups contains monoids, bands, and Clifford semigroups. Those ε −semigroup S for which $\varepsilon(S)$ is in the center of S have a structure theorem of strong sort. First, a definition: An ε −semigroup S is called a *partial monoid* if (PM5) ε_x is central (for all $x \in S$)

If S is a partial monoid, then an ε −subsemigroup of S is called a subpartial monoid of S. A partial monoid homomorphism is defined similarly.

In view of (PM2), we have $\varepsilon(S)$ is an idempotent ε −subsemigroup (subpartial monoid) of S whenever S is an ε −semigroup (partial monoid). In particular (by PM5), if S is a partial monoid, then $\varepsilon(S)$ is a commutative semigroup of idempotents, i.e. a semilattice (with the usual partial ordering $\varepsilon_x \le \varepsilon_y$ iff $\varepsilon_x\, \varepsilon_y = \varepsilon_x$)

Theorem 1.1 *The following two statements about a semigroup S are equivalent.*
(A) S is a partial monoid.
(B) S is a strong semilattice of monoids.

Remark
The above theorem shows that if S is a partial monoid, then S is a strong semilattice of monoids

$$S = \left[\varepsilon(S), S_{\varepsilon_x}, \varphi_{\varepsilon_x,\varepsilon_y}\right]$$

we have for ε_x in the semilattice $\varepsilon(S)$, S_{ε_x} is the maximal monoid $\{y \in S : \varepsilon_y = \varepsilon_x\}$ with the identity ε_x and for $\varepsilon_x \ge \varepsilon_y$ (i.e. $\varepsilon_x\, \varepsilon_y = \varepsilon_y$)

$$\varphi_{\varepsilon_x,\varepsilon_y} : S_{\varepsilon_x} \to S_{\varepsilon_y}, a \mapsto a\varepsilon_y$$

Conversely, if S is a strong semilattice of monoids $S = [\Upsilon, S_\alpha, \psi_{\alpha,\beta}]$, then S is a partial monoid with $\varepsilon -$ operation for $x \in S$, say $x \in S_\alpha$, $\varepsilon_x = e_\alpha$ where e_α is the identity of the monoid S_α. In [5] some topological and categorical aspects of partial monoids (not needn't in our present work) are obtained as well as a representation theorem says that every partial monoid S is empeddablle in a certain partial monoid of partial mappings in analogy with the same sort of theorem known for strong semilattices of groups i.e. Clifford semigroups (see [1], where they are called partial groups) and for strong semilattices of rings (see[4], where they are called partial rings). In [5] examples are given to show that :

For an $\varepsilon -$ semigroup $S, \varepsilon(S)$ may be a proper subset of $E(S)$, i.e. an idempotent in S may not be a partial identity.

There may exist different $\varepsilon -$ semigroup structures on the same semigroup S.

There may exist an $\varepsilon -$ semigroup S which is not a partial monoid (and hence not a monoid).

Non trivial partial monoids exists, i.e. partial monoids which are not monoids (These are introduced, in particular by partial mappings (from sets to monoids). Less trivially, every partial monoid S is embeddablle in a certian partial monoid of partial mappings [5, Theorem 3.4].

As shown in [5], it is easy to observe that the class of partial monoids is a variety (Ω, E) of algebras for some generator domain Ω, and set of equations E, whence free partial monoids exist. In the next section, we introduce explicit construction of free partial monoids and develop the basic properties and characterization of them.

Our references are , in semigroups, in general , [15] , [16], [17], [24], in semilattices of monoids [5],[25], and in free monoids and codes [9], [22] .

2 Free partial monoids. Construction and basic characterizations

Let A be a non empty (not necessarily finite) set. For every nonempty finite subset B of A, let ε_B denote the identity element of the free monoid B^* on B. In other words, ε_B stands for the empty word in B^*. There exists a natural embedding (satisfying the usual universal property)

$$\eta_B: B \to B^*$$

Actually, η_B sends each element b in the set B to the word in B^* that consists only of one alphabet b.

Let $\varepsilon(A) = \{\varepsilon_B : B \text{ is a non empty finite subset of } A\}$. Partially ordered $\varepsilon(A)$ by

$$\varepsilon_C \le \varepsilon_B \text{ if and only if } B \subset C.$$

Then $\varepsilon(A)$ with $\le$ is clearly a semilattice, whence the greatest lower bound of any $\varepsilon_B, \varepsilon_C \in \varepsilon(A)$ is given by

$$\varepsilon_B \wedge \varepsilon_C = \varepsilon_{B \cup C}.$$

Equivalently, $\varepsilon(A)$ is a commutative band with binary operation given by

$$\varepsilon_B \varepsilon_C = \varepsilon_{B \cup C} \text{ for all } \varepsilon_B, \varepsilon_C \in \varepsilon(A),$$

and we have for all $\varepsilon_B, \varepsilon_C \in \varepsilon(A)$

$\varepsilon_B \varepsilon_C = \varepsilon_C$ if and only if $\varepsilon_C \leq \varepsilon_B$ if and only if $B \subset C$.

For $\varepsilon_B \geq \varepsilon_C$ in $\varepsilon(A)$,(i.e. $B \subset C$), we define a mapping

$\varphi_{\varepsilon_B,\varepsilon_C}: B^* \to C^*$

as follows: For any non empty word $w \in B^*$, say $w = \eta_B b_1 \ldots \eta_B b_n (b_i \in B)$, we set

$\varphi_{\varepsilon_B,\varepsilon_C} w = \eta_C b_1 \ldots \eta_C b_n$.

For the empty word ε_B of B^*, we set

$\varphi_{\varepsilon_B,\varepsilon_C} \varepsilon_B = \varepsilon_C$ (the empty word in C^*).

We observe that $\varphi_{\varepsilon_B,\varepsilon_C}$ is a well defined mapping (since $B \subset C$) and clearly a monoid homomorphism. Actually, $\varphi_{\varepsilon_B,\varepsilon_C}$ is a monoid monomorphism. It is also easy to see that $\varphi_{\varepsilon_B,\varepsilon_B}$ is the identity automorphism of B^* and that

$\varphi_{\varepsilon_C,\varepsilon_D} \cdot \varphi_{\varepsilon_B,\varepsilon_C} = \varphi_{\varepsilon_B,\varepsilon_D}$

for all $\varepsilon_B, \varepsilon_C, \varepsilon_D$ in $\varepsilon(A)$, satisfying $\varepsilon_B, \geq \varepsilon_C \geq \varepsilon_D$. Summing up, we have a strong semilattice of monoids

$FPM(A) = \langle \varepsilon(A), B^*, \varphi_{\varepsilon_B,\varepsilon_C} \rangle$.

Whence,

$FPM(A) = \bigcup_{\varepsilon_B \in \varepsilon(A)} B^*_{\varepsilon_B}$

is a partial monoid, with operation (extending the operation of $B^*_{\varepsilon_B}$, $\varepsilon_B \in \varepsilon(A)$ given by , for any two elements in $FPM(A)$, say

$w_B = \eta_B b_1 \ldots \eta_B b_n \in B^*$ and $w_C = \eta_C c_1 \ldots \eta_C c_m \in C^*$

we have

$w_B \cdot w_C = \varphi_{\varepsilon_B,\varepsilon_{B\cup C}} w_B \cdot \varphi_{\varepsilon_C,\varepsilon_{B\cup C}} w_C$

$= \eta_{B\cup C} b_1 \ldots \eta_{B\cup C} b_n \eta_{B\cup C} c_1 \ldots \eta_{B\cup C} c_m \in (B \cup C)^*$

We define a mapping

$\eta: A \to FPM(A) = \bigcup_{\varepsilon_B \in \varepsilon(A)} B^*_{\varepsilon_B}$

by

$a \mapsto \eta_{\{a\}} a$

i.e. sending each element $a \in A$ to the word in $\{a\}^*$ consisting of one letter, namely a. Now, let S be any partial monoid and let $\psi: A \to S$ be any mapping of sets . Define

$\bar{\psi}: FPM(A) \to S$

as follows: Let $w \in FPM(A)$, be arbitrary. Then, there is a (unique) nonempty finite subset B of A, say $B = \{b_1, b_2, \ldots, b_r\}$ and a unique (may be empty) subset $\{b_{i_1}, b_{i_2}, \ldots, b_{i_n}\}$ of B, with

$$w = w_B = \eta_B b_{i_1} \ldots \eta_B b_{i_n}.$$

Define

$$\bar{\psi} w = \psi b_{i_1} \ldots \psi b_{i_n} \prod_{i=1}^{r} \varepsilon_{\psi b_i}$$

where $\varepsilon_{\psi b_i}$ is the partial identity of the element ψb_i in the partial monoid S. It follows that for each $w_B \in B^*$ we have

$$\varepsilon_{(\bar{\psi} w_B)} = \varepsilon_{\psi b_1} \varepsilon_{\psi b_2} \ldots \varepsilon_{\psi b_r} = \prod_{i=1}^{r} \varepsilon_{\psi b_i}.$$

Identifying $a \in A$ with $\eta_{\{a\}} a$ and $\varepsilon_{\{a\}}$ with ε_a, then the identity ε_B, viewed as the empty word in B^*, may be written

$$\varepsilon_B = \varepsilon_{\eta_{\{b_1\}} b_1} \varepsilon_{\eta_{\{b_2\}} b_2} \ldots \varepsilon_{\eta_{\{b_r\}} b_r}$$

$$= \varepsilon_{b_1} \varepsilon_{b_2} \ldots \varepsilon_{b_r} = \prod_{i=1}^{r} \varepsilon_{b_i}$$

By the definition of $\bar{\psi}$, $\bar{\psi} \varepsilon_B = \prod_{i=1}^{r} \varepsilon_{\psi b_i}$, and since $\varepsilon_{(w_B)} = \varepsilon_B$, it follows that

$$\varepsilon_{\bar{\psi} w_B} = \bar{\psi}\left(\varepsilon_{(w_B)}\right).$$

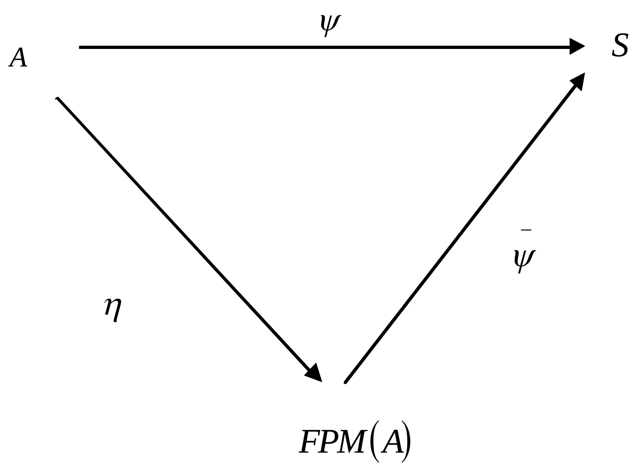
Clearly, $\bar{\psi}$ is a partial monoid homomorphism, with $\bar{\psi}\left(B^*_{\varepsilon_B}\right) \subset S_{\prod_{i=1}^{r} \varepsilon_{\psi b_i}}$, where $S_{\prod_{i=1}^{r} \varepsilon_{\psi b_i}}$ is the maximal monoid in S with identity $\bar{\psi}(\varepsilon_B) = \varepsilon_{\psi b_i}$.

For every $a \in A$, we have

$$\left(\bar{\psi} \eta\right)(a) = \bar{\psi}(\eta a) = \bar{\psi}\left(\eta_{\{a\}} a\right)$$

$$= \psi a \cdot \varepsilon_{\psi a} = \psi a \quad ,$$

whence, $\bar{\psi} \eta = \psi$, that is the diagram



Commutes.
If $\varphi: FPM(A) \to S$ is a partial monoid homomorphism with $\varphi\eta = \psi$, we have for any $w \in FPM(A)$, say $w = w_B = \eta_B b_{i_1} \ldots \eta_B b_{i_n} \in B^*$ (with $B = \{b_1, \ldots, b_r\}$),

$$\varphi w = \varphi\left(\eta_{\{b_{i_1}\}} b_{i_1} \ldots \eta_{\{b_{i_n}\}} b_{i_n} \cdot \varepsilon_B\right)$$
$$= \varphi\eta_{\{b_{i_1}\}} b_{i_1} \ldots \varphi\eta_{\{b_{i_n}\}} b_{i_n} \cdot \varphi(\varepsilon_B)$$
$$= \varphi\eta b_{i_1} \ldots \varphi\eta b_{i_n} \cdot \varphi(\varepsilon_B)$$
$$= \psi b_{i_1} \ldots \psi b_{i_n} \cdot \varphi(\varepsilon_B)$$

since φ is a partial monoid homomorphism, we have $\varphi\left(B^*_{\varepsilon_B}\right) \subset S_{\varphi(\varepsilon_B)}$. Now,

$$\varepsilon_B = \varepsilon\left({}_{\eta_{\{b_1\}}} b_1 \varepsilon_{\eta_{\{b_2\}}} b_2 \ldots \varepsilon_{\eta_{\{b_r\}}} b_r\right)$$

thus

$$\varphi(\varepsilon_B) = \varepsilon_{\varphi\left({}_{\eta_{\{b_1\}}} b_1 \varepsilon_{\eta_{\{b_2\}}} b_2 \ldots \varepsilon_{\eta_{\{b_r\}}} b_r\right)}$$
$$= \varepsilon_{(\varphi\eta b_1 \ldots \varphi\eta b_r)}$$
$$= \varepsilon_{(\psi b_1 \ldots \psi b_r)} = \varepsilon_{\psi b_1} \ldots \varepsilon_{\psi b_r}$$
$$= \prod_{i=1}^{r} \varepsilon_{\psi b_i}.$$

Thus,

$$\varphi w = \psi b_{i_1} \ldots \psi b_{i_n} \cdot \varphi(\varepsilon_B)$$
$$= \psi b_{i_1} \ldots \psi b_{i_n} \cdot \prod_{i=1}^{r} \varepsilon_{\psi b_i}$$
$$= \bar{\psi} w.$$

Hence $\varphi = \bar{\psi}$. Therefore $\bar{\psi}$ is the unique partial monoid homomorphism such that $\bar{\psi}\eta = \psi$, and we have proved ,

Theorem 2.1 *For any non empty set* A, *the partial monoid* $FPM(A)$ *is (up to an isomorphism) the free partial monoid on* A.

Remakes

In the free partial monoid $FPM(A)$, the effect of multiplying a "word" w_B by an $\varepsilon_C \in \varepsilon(A)$ (for some finite subsets $B, C \subset A$) is nothing but transforming $w_B \in B^*$ to the word $w_{B\cup C} \in (B \cup C)^*$ having the same string of alphabets as w_B. In particular if $b \in B$, then $\eta_B b \in B^*$ may be viewed as $\eta_{\{b\}} b.\varepsilon_B$. (Hence $\eta_{\{b\}} b \in \{b\}^*$) As we identify $\eta_{\ \{b\}} b = \eta b$ with b, we may write $\eta_B b = b\varepsilon_B$. Thus if $w_B = \eta_B b_{i_1} ... \eta_B b_{i_n}$ is a word in B^*, we may write

$$w_B = \eta b_{i_1} ... \eta b_{i_n} \varepsilon_B$$
$$= b_{i_1} ... b_{i_n} \varepsilon_B.$$

It follows that each (non empty) word w in $FPM(A)$ say $w = w_B$, for some non empty finite subset B of A, has a unique representation as product of alphabets from $B \subset A$ with ε_B.

In the rest of this section we give some characterizations of free partial monoids analogous to the known characterizations [22] of free monoids. We start with a definition.

Let M be a partial monoid (i.e. strong semilattice $[\varepsilon(A), M_{\varepsilon_a}, \varphi_{\varepsilon_a, \varepsilon_b}]$ of monoids). We call a subset A of M a set of partial generators of (or partially generates) M if for every $b \in M$, with $b \neq \varepsilon_b$, there is a finite set $\{a_1, ..., a_r\} \subset A$ such that

$$b = x_1 x_2 ... x_n \varepsilon_{\prod_{i=1}^{r} a_i}$$

with $x_i,\ i = 1,2, ..., n$ (possibly not all distinct) are elements of $\{a_1, ..., a_r\}$.

Theorem 2.2 *Let* A *be a (non empty) set,* M *a partial monoid and let* $i: A \to M$ *be an injection onto a set of partial generators of* M. *The following two statements are equivalent:*

(a) M is free on $i(A)$.

(b) For any partial monoid M' and map $\varphi: A \to M'$, there is a unique homomorphism of partial monoids $\bar{\varphi}: M \to M'$ such that $\varphi = \bar{\varphi} i$.

Proof. $(a) \Rightarrow (b)$. Let $\eta: i(A) \to M$ be the natural embedding (as in Theorem 2.1). Define $\varphi': i(A) \to M'$ by $i(a) \mapsto \varphi(a)$, where $\varphi: A \to M'$ is a given map. φ' is a well defined map, since i is injection. By Theorem 2.1, there exists a unique partial monoid homomorphism $\bar{\varphi}': M \to M'$ such that $\varphi' = \bar{\varphi}' \circ \eta$. We have $\varphi'(i(a)) = \varphi(a),\ (a \in A)$. Thus

$$\varphi(a) = \varphi'(i(a)) = \bar{\varphi}' \circ \eta(i(a))$$
$$= \bar{\varphi}'(i(a)\varepsilon_{\eta i(a)})$$

$= \bar{\varphi}'(i(a))\varepsilon_{\bar{\varphi}'\eta i(a)} = \bar{\varphi}' i(a).$

Therefore, $\varphi = \bar{\varphi}' i$.

$(b) \Rightarrow (a)$. Let $M' = [i(A)]^*$ be the free partial monoid on $i(A)$, and let $\eta: i(A) \to M'$ be the natural embedding. Let $\varphi: i(A) \to M$ be the inclusion map. As in the proof $(a) \Rightarrow (b)$, (since M' is free on $i(A)$), there is a partial monoid homomorphism

$\Phi: FPM(i(A)) = M' \to M$

such that $\Phi\eta = \varphi$. That is $\Phi\eta(i(a)) = \varphi(i(a)) = i(a)$, $(a \in A)$, and Φ is the identity on the partial generators of M'. Let $\psi: A \to M'$ be given by

$\psi = \eta i: A \overset{i}{\to} i(A) \overset{\eta}{\to} M'.$

By (b) there is a partial monoid homomorphism $\Psi: M \to M'$ such that $\Psi i = \psi$. For $i(a) \in M$, we have

$\Psi(i(a)) = \Psi i(a) = \psi(a) = \eta i(a).$

Let $x \in M$, $(x \neq \varepsilon_x)$ be arbitrary, say $x = ia_{j_1} ia_{j_2} \ldots ia_{j_n} \varepsilon_{\prod_{l=1}^{r} ia_l}$ with $a_{j_k} \in \{a_1, \ldots, a_r\}, k = 1,2,\ldots,n$ (observe that $i(A)$ partially generates M), and $\varepsilon_x = \varepsilon_{\prod_{l=1}^{r} ia_l}$. We have $\Psi x \in M'$ and so $\psi x \in B^*$ for some finite set $B = \{b_1, \ldots, b_r\} \subset i(A)$. Now clearly $\varepsilon_{\Psi x} = \varepsilon_{\prod_{i=1}^{r} ia_l}$

which gives

$\varepsilon_{\Phi\Psi x} = \varepsilon_{\overset{r}{\underset{l=1}{\Phi}} \eta ia_l} = \varepsilon_{\prod_{i=1}^{r} ia_l} = \varepsilon_x.$

Therefore,

$\Phi\Psi x = \Phi(\Psi ia_{j_1} \ldots \Psi ia_{j_n})\Psi\varepsilon_x$

$= \Phi(\eta ia_{j_1} \ldots \eta ia_{j_n})\varepsilon_{\Psi x}$

$= \Phi\eta ia_{j_1} \ldots \Phi\eta ia_{j_n} \varepsilon_{\Phi\Psi x}$

$= \varphi(ia_{j_1}) \ldots \varphi(ia_{j_n})\varepsilon_{\Phi\Psi x}$

$= ia_{j_1} \ldots ia_{j_n} \varepsilon_x = x.$

Thus $\Phi\Psi = id_M$ (the identity map $M \to M$). Likewise, for $x \in M'$, say $x = \eta ia_{j_1} \ldots \eta ia_{j_n} \varepsilon_x$ where ε_x is the identity of the maximal monoid, say B^* in M', for some finite subset $B = \{ia_1, \ldots, ia_r\}$ of $i(A)$. Thus $\varepsilon_x = \varepsilon_B = \varepsilon_{\eta ia_1} \ldots \varepsilon_{\eta ia_r} = \varepsilon_{\eta ia_1 \ldots \eta ia_r} = \varepsilon_{ia_1 \ldots ia_r}$. We have

$\Psi\Phi(x) = \Psi(\Phi\eta ia_{j_1}...\Phi\eta ia_{j_n}\Phi\varepsilon_x)$
$= \Psi(ia_{j_1}...ia_{j_n}\varepsilon_{\Phi(x)})$
$= \Psi ia_{j_1}...\Psi ia_{j_n}\varepsilon_{\Psi\Phi x}$
$= \eta ia_{j_1}...\eta ia_{j_n}\varepsilon_x = x.$

Thus $\Psi\Phi = id_{M'}$. It follows that M is isomorphic to M' and the proof is complete.

Given a partial monoid M, we set $S = M - \varepsilon(M)$ and
$A = \{x \in S - S^2\colon \varepsilon_x$ is maximal in $\varepsilon(M)\}$. Then we have:

Theorem 2.3 *Let M be a partial monoid and let A be the subset of Mdefined as above. The following two statements are equivalent:*
(a) M is free on A
(b) For each $x \in M$ with $x \neq \varepsilon_x$, there exists a unique finite set $\{a_1, ..., a_r\} \subset A$ such that $\varepsilon_x = \prod_{i=1}^{r} \varepsilon_{a_i}$ and x has a unique factorization

$x = x_1...x_m\varepsilon_x$

with $\{x_1, ..., x_m\} \subset \{a_1, ..., a_r\}$

Proof. $(a) \Rightarrow (b)$. Follows from the definition of a free partial monoid and the property of A.
$(b) \Rightarrow (a)$. Let M' be the free partial monoid on A, and let $\eta\colon A \to M'$ be the natural embedding. Let

$\varphi'\colon A \to M$

be the inclusion map, that is $\varphi'(a) = a\varepsilon_a = a \quad (a \in A)$. By the universal properly (cf. Theorem 2.1), there exists a unique partial monoid homomorphism

$\bar{\varphi}'\colon M' \to M$

such that $\varphi' = \bar{\varphi}' \circ \eta$. We have $\varphi'\colon A \to M$ is the inclusion onto the partial generators of M. Define

$\psi\colon M \to M'$

by

$\psi(x_1x_2...x_m\varepsilon_x) = \eta x_1\eta x_2...\eta x_m \prod_{i=1}^{r} \varepsilon_{\eta x_i}.$

Clearly ψ is partial monoid homomorphism and $\eta = \psi\varphi'$. Let $b \in M$. By (b), there exists a unique set $\{a_1, ..., a_r\} \subset A$ such that $\varepsilon_b = \prod_{i=1}^{r} \varepsilon_{a_i} = \varepsilon_{\prod_{i=1}^{r} a_i}$ and b has a unique factorization

$b = b_1 b_2 \ldots b_n \varepsilon_b = b_1 b_2 \ldots b_n \varepsilon_{\prod_{i=1}^{r} a_i}$

with $\{b_1, b_2, \ldots, b_n\} \subset \{a_1, \ldots, a_r\}$. As in the proof $(b) \Rightarrow (a)$ of theorem 2.2 we can show $\varepsilon_{\bar{\varphi}'\psi b} = \varepsilon_b$. Thus we have

$\bar{\varphi}'\psi b = \bar{\varphi}'(\psi\varphi' b_1 \ldots \psi\varphi' b_n \varepsilon_{\psi b})$

$= \bar{\varphi}'(\eta b_1 \ldots \eta b_n \varepsilon_{\psi b})$

$= \bar{\varphi}'\eta b_1 \ldots \bar{\varphi}'\eta b_n \varepsilon_{\bar{\varphi}'\psi b}$

$= \varphi' b_1 \ldots \varphi' b_n \varepsilon_b = b_1 b_2 \ldots b_n \varepsilon_b = b.$

Thus $\bar{\varphi}'\psi = id_M$. Let $b \in M'$, say $b \in B^*$ for some finite $B \subset A$, there exist $b_1, b_2, \ldots, b_n \in A$ with

$b = \eta b_1 \ldots \eta b_n \varepsilon_b.$

We have

$\psi\bar{\varphi}' b = \psi\left(\varphi'\eta b_1 \ldots \varphi'\eta b_n \varepsilon_{\bar{\varphi}' b}\right)$

$= \psi\left(\varphi' b_1 \ldots \varphi' b_n \varepsilon_{\bar{\varphi}' b}\right)$

$= \psi\varphi' b_1 \ldots \psi\varphi' b_n \varepsilon_{\psi\bar{\varphi}' b}$

$= \eta b_1 \ldots \eta b_n\, \varepsilon_{\psi\bar{\varphi}' b}$

$= \eta b_1 \ldots \eta b_n \varepsilon_b = b.$

Thus $\psi\bar{\varphi}' = id_{M'}$. Therefore $M = M'$, and so, M is free on A.

We may conclude directly the following

Corollary 2.1 *Let M be a partial monoid satisfying one of the two equivalent conditions of Theorem 2.3*

(i) For every $x \in M, M_{\varepsilon_x}$ is free monoid with finite set of (free) generators $\{a_1\varepsilon_x, \ldots, a_r\varepsilon_x\}$ where $\{a_1, \ldots, a_r\} \subset A$ is the unique set such that $\varepsilon_x = \prod_{i=1}^{r} \varepsilon_{a_i}$. In particular, for every $x \in A, M_{\varepsilon_x}$ is cyclic with one generator a, where a is the unique element in A such that $\varepsilon_x = \varepsilon_a$

(ii) Every $\varepsilon_x \in \varepsilon(A) = \varepsilon(M)$ has a unique factorization $\varepsilon_x = \prod_{i=1}^{r} \varepsilon_{a_i}, a_i \in A$. In particular if $\varepsilon_x = \prod_{i=1}^{r} \varepsilon_{a_i}$ and $\varepsilon_y = \prod_{j=1}^{s} \varepsilon_{b_j}$, $a_i, b_j \in A$ then

$\varepsilon_x = \varepsilon_y \quad \text{iff} \quad \{a_i : i = 1, \ldots, r\} = \{b_j : j = 1, \ldots, s\}$

(iii) For every $\varepsilon_a \geq \varepsilon_b$,

$$\varphi_{\varepsilon_a,\varepsilon_b}: M_{\varepsilon_a} \to M_{\varepsilon_b}$$

is a monomorphism of monoids
(iv) For every $b \in S - S^2$, there exists a unique $a \in A$ such that $a\varepsilon_b = b$
(v) For every $b \in S - S^2$, we have $b\varepsilon_x \in S - S^2$, for every $\varepsilon_x \in \varepsilon(A)$

3 Equidivisability and Codes in free partial monoids

In this section we define equidivisibility in partial monoids and develope results analogous to the results in [19] concerning equidivisibility in monoids.
A partial monoid M is called equidivisible if for every $a, b, c, d \in M - \varepsilon(M)$

$$ab = cd$$

implies either $ae = cue, ube = de$ for some $u \in M$ or $ave = ce, be = vde$ for some $v \in M$ where $e = \varepsilon_{ab} = \varepsilon_{cd}$

Lemma 3.1 *A partial monoid M is equidivisible if and only if every maximal monoid M_{ε_a} is equidivisible*

Proof. Suppose that M is equidivisible. Choose any $e \in \varepsilon(M)$ and let $a, b, c, d \in M_e - \{e\}$ be such that $ab = cd$. Since M is equidivisible and $\varepsilon_{ab} = \varepsilon_{cd} = \varepsilon_a = \varepsilon_b = \varepsilon_c = \varepsilon_d = e$, we have either $ae = cue, ube = de$ for some $u \in M$ or $ave = ce, be = vde$ for some $v \in M$. That is either $a = cu, ub = d$ for some $u \in M$ or $av = c, b = vd$ for some $v \in M$. Setting $u' = ue$ and $v' = ve$, we have

$$\varepsilon_{u'} = \varepsilon_u e = e \text{ and } \varepsilon_{v'} = \varepsilon_v e = e$$

and so $u', v' \in M_e$. Then we have either

$$a = cu = (ce)u = c(ue) = cu',$$
$$u'b = (ue)b = u(be) = ub = d$$

for some $u' \in M_e$ or

$$av' = a(ve) = (ae)v = av = c,$$
$$b = vd = v(de) = (ve)d = v'd$$

for some $v' \in M_e$. Thus M_e is equidivisible

Lemma 3.2 *Every free partial monoid is equidivisible*

Proof. Since a free partial monoid is a strong semilattice of (its maximal) free monoids (c.f. Theorem 2.1) and every free monoid is clearly equidivisible [19,Ch5,Cor1.6 or 1.7], the result obtains from (the if part of) Lemma 3.1

Theorem 3.1 *Let M be a partial monoid with non trivial maximal monoids and let $S = M - \varepsilon(M)$. Suppose the following three conditions hold*
(i) $\varphi_{e,f}$ is a monomorphism for all $e \geq f$ in $\varepsilon(M)$
(ii) For every $b \in S - S^2$, there exists a unique $a \in A$ such that $a\varepsilon_b = b$
(iii) For every $x \in M$, there exists a unique (finite non empty) set $\{a_1, \ldots, a_r\} \subset A$ such that

$$\varepsilon_x = \prod_{i=1}^{r} \varepsilon_{a_i}$$

where $A = \{a \in S - S^2 \colon \varepsilon_a \text{ is maximal}\}$.

Then M is free on A if and only if M is equidivisible and $\bigcap_{n\in\mathbb{N}} S^{2n-1} = \emptyset$.

Proof. If M is free on A, then M is equidivisible by Lemma 3.2 and hence by Lemma 3.1 every maximal monoid in M is (free by definition) equidivisible. It follows that $\bigcap_{n\in\mathbb{N}} S_e^{2n-1} = \emptyset$ for every $e \in \varepsilon(M)$. Since $S = \bigcup_{e\in\varepsilon(M)} S_e$ and $\{S_e \colon e \in \varepsilon(M)\}$ are pairwise disjoint, we have

$$\bigcap_{n\in\mathbb{N}} S^{2n-1} = \bigcap_{n\in\mathbb{N}} \left(\bigcup_{e\in\varepsilon(M)} S_e\right)^{2n-1} = \bigcap_{n\in\mathbb{N}} \left(\bigcup_{e\in\varepsilon(M)} S_e^{2n-1}\right)$$
$$\subset \bigcup_{e\in\varepsilon(M)} \left(\bigcap_{n\in\mathbb{N}} S_e^{2n-1}\right) = \emptyset.$$

Conversely, suppose that M is equidivisible and $\bigcap_{n\in\mathbb{N}} S^{2n-1} = \emptyset$. If $a \in S$, is idempotent, we have $a = a^n$ for all $n \in \mathbb{N}$ which implies $a \in \bigcap_{n\in\mathbb{N}} S^{2n-1} = \emptyset$ a contradiction. Thus we have (iv) $\varepsilon(M) = E(M)$.

By assumption, every structure arrow $\varphi_{e,f}$ ($e \geq f$) is a monomorphism. Equivalently, we have :
(v) For every $x, y \in M, x \notin \varepsilon(M)$ implies $x\varepsilon_y \notin \varepsilon(M)$. Now let $a \in S$ be a unit, that is there exists $a^{-1} \in S$ such that $aa^{-1} \in \varepsilon(M)$ and hence by (iv) $a^{-1}a \in \varepsilon(M)$ say $a^{-1}a = e \in \varepsilon(M)$. We have by (v) $ae \notin \varepsilon(M)$. Thus $ae \in S$ and

$$ae = aa^{-1}aa^{-1}a\ldots \in \bigcap_{k\in\mathbb{N}} S^{2k-1} = \emptyset$$

a contradiction. Hence the set of units is $\varepsilon(M)$. Thus (v) may be refined as follows:

(vi) For all $x, y \in M, xy \in \varepsilon(M)$ if and only if $x, y \in \varepsilon(M)$. Thus we have a descending chain $S \supseteq S^2 \supseteq S^3 \supseteq \ldots \supseteq S^k \supseteq S^{k+1} \supseteq \ldots$.If the chain statilizes, say $S^k = S^{k+1}$ then $S^k = \bigcap_{n\in\mathbb{N}} S^n = \emptyset$ which gives (by(vi)) $S = \emptyset$ and M is the free partial monoid on the empty set . If $S \supset S^2 \supset S^3 \supset \ldots \supset S^k \supset S^{k+1} \supset \ldots$ Is strictly descending, then for every $m \in S$, there exists k such that $m \in S^k - S^{k+1}$ (otherwise $m \in \bigcap_{n\in\mathbb{N}} S^n = \emptyset$ a contradiction). Thus $m = x_1x_2\ldots x_k$ with $x_i \in S - S^2, 1 \leq i \leq k$. If $m = x_{i_1}x_{i_2}\ldots x_{i_k} = x_{j_1}x_{j_2}\ldots x_{j_l}$ are two factorizations of m, then (using equidivisibility and (i)) we can show by induction on $n = \min(l, k)$ that $k = l$ and $x_{i_1}\varepsilon_m = x_{j_1}\varepsilon_m, \ldots, x_{i_k}\varepsilon_m = x_{j_k}\varepsilon_m$. It follows by (iii) that for each $i, 1 \leq i \leq k$ there exists a unique $a_i \in A, 1 \leq i \leq k$ such that $a_i \varepsilon_{x_i} = x_i$. Hence m has a unique factorization

$$m = a_1 a_2 \ldots a_k \prod_{i=1}^{k} \varepsilon_{x_i}$$
$$= a_1 a_2 \ldots a_k \varepsilon_m \qquad (a_i \in A, 1 \leq i \leq k).$$

Lemma 3.3 *Let M be a full subpartial monoid of $FPM(A)$, and let $M^+ = M - \varepsilon(M)$, then $M^+ - (M^+)^2$ generates M and contained in every set that generates M*

Proof. By the above theorem we have $\bigcap_{n\in\mathbb{N}} S_M^n = \emptyset$ where $S_M = M^+$. The result obtains .

Let M be as above and let $C = \{a \in M^+ - (M^+)^2 : \varepsilon_a \text{is maximal}\}$

C is a (possibly empty) set contained in every set generating M. If C partially generates M it is called *a base* for M in the sense that, $m \in M$ implies $m = a_1 a_2 \ldots a_p \varepsilon_m$ for some $a_i \in C \quad (1 \leq i \leq p)$.

Lemma 3.4 *If M has a base C, then*
For every $m \in M^+ - (M^+)^2$ there exists a (not necessery unique) element $a \in C$ such that $a\varepsilon_m = m$
(ii) For every $m \in M^+ - (M^+)^2$ and every $\varepsilon \in \varepsilon(M)$, $m\varepsilon \in M^+ - (M^+)^2$

Proof. (i) By assumption there exist $a_1, a_2, \ldots, a_p \in C$ such that $m = a_1 a_2 \ldots a_p \varepsilon_m$. We have $a_i \varepsilon_m \in M^+$,$(1 \leq i \leq p)$ (c.f.(v)). Thus $m \in M^+ - (M^+)^2$ implies $p = 1$ and $m = a_1 \varepsilon_m$.

(ii) By assumption and (v) we have $m\varepsilon \in (M^+)$.If $m\varepsilon \in (M^+)^2$, we would have
$$m\varepsilon = x_1 x_2 \quad \text{for some } x_i \in M^+, i = 1,2.$$

Write $m = a\varepsilon_m, a \in C$ (by (i)) we would have
$$a\varepsilon_m = a_1 a_2 \ldots a_p \varepsilon_{a_m}$$

which implies by maximality $\varepsilon_a = \varepsilon_{a_i}$ for all $(1 \leq i \leq p)$. Thus
$$\varphi_{\varepsilon_a, \varepsilon_{a_m}} a = \varphi_{\varepsilon_a, \varepsilon_{am}} a_1 \ldots \varphi_{\varepsilon_a, \varepsilon_{a_m}} a_p$$

which implies that $a = a_1 a_2 \ldots a_p$ (since φ is monomorphism) a contradiction.

Lemma 3.5 *M full subpartial monoid of $FPM(A)$ has a base if and only if $m \in M^+ - (M^+)^2$ implies there exists $a \in M^+ - (M^+)^2$, ε_a maximal and $m = a\varepsilon_m$*

Proof. Follows from Lemma 3.3 and Lemma 3.4.

Theorem 3.2 *Let M be a full subpartial monoid of a free partial monoid $FPM(A)$.*

Suppose M has a base C. The following conditions are equivalent:
(a) M is free
(b) For every $w \in FPM(A), Mw \cap M \neq \emptyset$ and $wM \cap M \neq \emptyset$ imply $w \in M$
(c) For every $w \in FPM(A), Mw \cap M \cap wM \neq \emptyset$ implies $w \in M$.

Proof. $(a) \Rightarrow (b)$. Assume $m_1 w \in M$ and $wm_2 \in M$ for some $m_1, m_2 \in M$. Thus $m_1(wm_2) = (m_1 w)m_2 \in M$. Since M is equidivisible (by Lemma 3.2), either

$$m_1 \varepsilon_u = m_1 wm\varepsilon_u \text{ where } \varepsilon_u = \varepsilon_{(m_1 wm_2)}, \text{ for some } m \in M$$

or

$$m_1 w\varepsilon_u = m_1 m' \varepsilon_u, \text{ for some } m' \in M$$

In the first case by cancellation in FPM(A)

$$\varepsilon_u = wm\varepsilon_u$$

and so $wm = \varepsilon_{wm}$ (cf.(v)) which implies $w = \varepsilon_w$ and $m = \varepsilon_m$ and thus $w \in M$ since M is wide. In the second case $w\varepsilon_u = m'\varepsilon_u$. Thus for some finite set of free generators $s_1 ... s_n \in FPM(A)$ we must have

$$w = s_1 ... s_n \varepsilon_w \text{ and } m' = s_1 ... s_n \varepsilon_{m'}.$$

Since M is free and $m' \in M$ the free generators of m' in M are uniquely given in terms of $s_1 ... s_n$. In particular $s_1 ... s_n \in M$. Whence $w = s_1 ... s_n \varepsilon_w \in M$.
$(b) \Rightarrow (c)$. Obvious
$(c) \Rightarrow (a)$. Suppose (c) holds but M is not free. There exists $w \in M^+$ with (at least) two different factorization, as product of partial generators from C. We choose such a $w \in M^+$ with minimal length in FPM(A)

$$w = c_{i_1} c_{i_2} ... c_{i_p} \varepsilon_w = c_{j_1} c_{j_2} ... c_{j_q} \varepsilon_w, \; c_{i_k}, c_{j_l} \in C \text{ for all } k, l \text{ and } c_{i_1} \neq c_{j_1}. \qquad (1)$$

By equividivisibility in FPM(A) either

$$c_{i_1} \varepsilon_w = c_{j_1} u\varepsilon_w \text{ for some } u \in FPM(A)$$

or

$$c_{i_1} v\varepsilon_w = c_{j_1} \varepsilon_w \text{ for some } v \in FPM(A)$$

we only deal with the first case as the other being similar

$c_{i_1} \varepsilon_w = c_{j_1} u\varepsilon_w$ and (1) yield

$$uc_{i_2} ... c_{i_p} \varepsilon_w = c_{j_2} ... c_{j_q} \varepsilon_w.$$

It follows that

$$c_{j_2} c_{j_3} ... c_{j_q} c_{i_1} \varepsilon_w = c_{j_2} ... c_{j_q} c_{j_1} u\varepsilon_w$$
$$= uc_{i_2} ... c_{i_p} c_{i_1} \varepsilon_w$$

may be written

$c_{j_2}c_{j_3}...c_{j_q}c_{i_1}\varepsilon_w = c_{j_2}c_{j_3}...c_{j_q}\varepsilon_w(u\varepsilon_u)$
$= (u\varepsilon_u)c_{i_2}...c_{i_p}c_{i_1}\varepsilon_w.$

By (c) $u \in M$ and $c_{i_1}\varepsilon_w = c_{j_1}u\varepsilon_w$ implies $u = \varepsilon_u$ otherwise $c_{i_1}\varepsilon_w$ (in $M^+ - (M^+)^2$ by Lemma 3.4 (ii)) would have a factorization $(c_{j_1}\varepsilon_w)(u\varepsilon_w) \in (M^+)^2$. Thus $c_{i_1}\varepsilon_w = c_{j_1}\varepsilon_w$. Cancelling in (1) by $c_{i_1}\varepsilon_w$ we have

$w' = c_{i_2}...c_{i_p}\varepsilon_w = c_{j_2}c_{j_3}...c_{j_q}\varepsilon_w$

are two distinct factorization of w' contradicting the minimality of w. To complete the proof we must show for every $m \in M$ there exists unique finite set $m_1, ..., m_p \in C$ with $\varepsilon_m = \prod_{i=1}^{p} \varepsilon_{m_i}$ and there also exists set $\{a_1, ..., a_p\} \subset \{m_1, ..., m_p\}$ such that

$m = a_1...a_p\varepsilon_m.$

It is sufficient to show that for every $a \in C, M_{\varepsilon_a}$ is cyclic (with generator b). This is clear otherwise we would have $b \in M_{\varepsilon_a}, b \neq a, b \in C$, and since ε_a is maximal, $\varepsilon_a = \varepsilon_{a_i}$ for some $a_i \in A$. Thus for some $m \neq n, b = a_1^n, a = a_1^m, w = a_1^{nm} \in M$ has two distinct factorizations $w = a^n = b^m$ a contradiction. Therefore, M is free.

A subset C of a free partial monoid $FPM(A)$ is called a partial code over A if C is the base of a free subpartial monoid M of $FPM(A)$, we write $M = FPM(C)$

Corollary 3.1 *Let C partial code over A, M free subpartial monoid of $FPM(A)$, with base C. Then for each $a \in M^+, M_{\varepsilon_a}$ is a free submonoid of $\left(FPM(A)\right)_{\varepsilon_a}$ with code (base)*

$C_{\varepsilon_a} = \{c\varepsilon_a : c \in C, \varepsilon_c \geq \varepsilon_a\}$

over B^*, where B is the set of free generators of $\left(FPM(A)\right)_{\varepsilon_a}$.

Proof. We have M_{ε_a} is a submonoid of $B^* = \left(FPM(A)\right)_{\varepsilon_a}$. Actually

$M_{\varepsilon_a} = M \cap B^*.$

Let $w \in B^*$ with $M_{\varepsilon_a}w \cap M_{\varepsilon_a} \cap wM_{\varepsilon_a} \neq \emptyset$. Thus, in particular we have

$Mw \cap M \cap wM \neq \emptyset$

and since M is free it follows by previous Lemma that $w \in M$. Whence $w \in M \cap B^* = M_{\varepsilon_a}$. The result follows by proposition 2.2 Ch.5 [19]

Corollary 3.2 *A subset C of $FPM(A)$ is a partial code over A if and only if*

(a) For every $c \in C, c = a^n$ for some $a \in A, n > 0$

(b) For every $c, d \in C$, say $c = a^n, d = b^m, c \neq d$ if and only if $a \neq b$.

Proof. Follows from the definition of a partial code and the properties of a free (sub) partial monoid.

Proposition 3.1 *Let M be a (wide) subpartial monoid of a free partial monoid $FPM(A)$, and let M have a base C. Then the following conditions are equivalent:*
(a) For every $w \in FPM(A), Mw \cap M \neq \emptyset$ implies $w \in M$
(b) $CFPM(A) \cap C = \emptyset$

[observe condition (b) doesn't necessarily hold in general, e.g. let M at level $a \in A$ have partial generators $a^2, a^5 \in C$ and $a^3 \notin C$. Therefore $CFPM(A) \cap C \neq \emptyset$ since $a^2a^3 = a^5 \in CFPM(A) \cap C$]

Proof. $(a) \Rightarrow (b)$. Assume (a) and suppose for some $c \in C, w \in FPM(A), cw = c_1$ for some $c_1 \in C$. Thus $Mw \cap M \neq \emptyset$, and so by (a) $w \in M$. It follows that $c_1 \in (M^+)^2$ a contradiction. $(b) \Rightarrow (a)$.Assume (b) and suppose $mw \in M$ for some $m \in M$ and $w \in FPM(A)$. We prove by induction on length of m (in FPM(A)) that $w \in M$. If $l(m) = 0$, then $m = \varepsilon_m$, which gives $w\varepsilon_m \in M$ that is $w\varepsilon_m = m'$ for some $m' \in M$. Therefore w and m' have the same partial generators as elements in FPM(A). Thus for some $a_1, \ldots, a_p \in A$ we have

$$m' = a_1 \ldots a_p \varepsilon_{m'},$$
$$w = a_1 \ldots a_p \varepsilon_w.$$

Since C partially generates M we must have $a_1 \ldots a_p = c_1 \ldots c_r$ for some $c_i \in C$ $(r \leq p)$. Thus $a_1 \ldots a_p \in M$ which gives $w \in M$. If $l(m) > 0$, then since C partially generates M, we have (since $mw \in M$)

$$c_{i_1} c_{i_2} \ldots c_{i_k} w \varepsilon_{wm} = c_{j_1} c_{j_2} \ldots c_{j_l} \varepsilon_{m'}$$

for some $m' = c_{j_1} c_{j_2} \ldots c_{j_l} \varepsilon_{m'}, c_{i_r}, c_{j_p} \in C \quad (1 \leq k, 1 \leq l)$. By (b) and equidivisibility in FPM(A) we have $c_{i_1} \varepsilon_{wm} = c_{j_1} u \varepsilon_{wm}$ for some $u \in$FPM(A) or $c_{i_1} v \varepsilon_{wm} = c_{j_1} \varepsilon_{wm}$ for some $v \in$FPM(A). We deal only with the first case, the second being similar. By maximality of elements of C, we have for some $a \in A, n > 0, c_{i_1} = a^n$. Thus

$$a^n \varepsilon_{wm} = c_{j_1} u \varepsilon_{wm}$$

this gives

$$c_{j_1} = a^p \quad \text{and} \quad u = a^r \varepsilon_u$$

for some $p > 0, r \geq 0$ and $p + r = n$. We have $a^n = a^p a^r$ with $a^n \in C, a^p \in\in C, a^r \in FPM(A)$. Thus by (b) we must have $p = n, r = 0$. This gives $u = \varepsilon_u$ and $c_{i_1} = c_{j_1}$. Therefore, (by cancellation in FPM(A))

$$c_{i_2} \ldots c_{i_k} w \varepsilon_{wm} = c_{j_2} \ldots c_{j_l} \varepsilon_{m'}.$$

Now the induction hypothesis implies $w \in M$.

If C is a partial code over the alphabet A, then the characterization given by Corollary 3.2 for C indicates that no word of C is a proper left factor of another word of C [otherwise we would have two words $a^n, a^m, n \neq m$ for some $a \in A$ which is impossible]. This is equivalent to the condition (b) of the above proposition

$CFPM(A) \cap C = \emptyset$

in which case C is called a *prefix* partial code. Similarly $\mathrm{FPM}(A)C \cap C = \emptyset$ always hold, for any partial code C, for which C is called a *suffix* partial code. Thus we have shown

Proposition 3.2 *Every partial code C over A is a biprefix partial code (i.e. prefix and suffix).*

The above two propositions give another simple characterization of partial codes.

Let C be a base for subpartial monoid M of $\mathrm{FPM}(A)$, then the following conditions are equivalent

Corollary 3.3 *(a) M is free i.e. C is a partial code over A*

(b) For every $w \in \mathrm{FPM}(A)$, $Mw \cap M \neq \emptyset$ implies $w \in M$

(c) For every $w \in \mathrm{FPM}(A)$, $wM \cap M \neq \emptyset$ implies $w \in M$

(d) $CFPM(A) \cap C = \emptyset$

(e) $\mathrm{FPM}(A)C \cap C = \emptyset$.

A partial code C over an alphabet A is called a maximal if $C \neq A$ and for every partial code C', $C \subset C'$ implies $C = C'$. Using characterization of partial codes in Corollary 3.2 , we have

Corollary 3.4 *A partial code C over A is maximal if and only if $C \cap \{a\}^* \neq \emptyset$ for every $a \in A$. equivalently C is maximal iff for every $a \in A$ there is some $c \in C$ with $\varepsilon_c = \varepsilon_a$.*

Example 3.3 *For each $n \geq 1$ the partial codes $C = \{a^n : a \in A\}$ are maximal*

Proposition 3.4 *Let C be a partial code over an alphabet A. The following conditions are equivalent*

(a) C is maximal

(b) For every $w \in \mathrm{FPM}(A)$, with ε_w is maximal w has a left factor in C or is a left factor of some $c \in C$

(c) For every $w \in FPM(A)$, with ε_w is maximal $wFPM(A) \cap FPM(C) \neq \emptyset$

(d) $\mathrm{FPM}(A)_{\max} = FPM(C)_{\max} P_{\max}$ where $P_{\max}$ is the set of all proper left factors of words in C, with maximal identities. [For each $a \in A$, $\left(FPM(A)\right)_{\varepsilon_a} = \left(FPM(C)\right)_{\varepsilon_a} P_{\varepsilon_a}$ where P_{ε_a} is the set of all proper left factors of words in C_{ε_a}.

Proof. $(a) \Rightarrow (b)$. Assume (a). Let $w \in$FPM(A) with ε_w is maximal . Thus $w = a^n$ for some $a \in A, n \geq 1$. Since C maximal $C \cap \{a\}^* \neq \emptyset$. Thus $c = a^m \in C$ for some $m \geq 1$. If $m \leq n$, c is clearly a left factor of w. If $m > n$ $a^m = a^n a^{m-n} = wa^{m-n} \in C$ and w is a left factor for $a^m = c \in C$. For $w = a_1 \ldots a_p \varepsilon_w$. By maximality of $C, c_i = a_i^{n_i} \in C$ for all i, and hence a_i is a left factor of c_i.
$(b) \Rightarrow (c)$. Assume (b). By induction on $l(w)$, every $w \in$FPM(A) with ε_w maximal , can be written as $w = c^n w'$, for some $n \geq 1, c \in C$, where w'has no left factor in $C, \varepsilon_{w'}$ maximal. Applying (b) to w', there exists $c' \in C$ such that $w'w'' = c'$ for some $w'' \in FPM(A)$. Thus $ww'' = c^n c' \in$FPM(C)
$(c) \Rightarrow (a)$. By (c), for every $a \in A, n \geq 1$,
$a^n FPM(A) \cap FPM(C) \neq \emptyset$.

Thus (for every $n \geq 1$) $a^n w_n = c_1 c_2 \ldots c_p \varepsilon_n$, for some $w_n \in FPM(A), c_i \in C$. By cancellation on left , $c_1 = a^m$ for some $m \geq 1$. Thus $a^m \in C$ and C is therefore, a maximal partial code.

Finally (d) is another formulation of (b).

A partial code satisfies one of the above conditions is called *complete.*

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