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Integral Solutions Of The Biquadratic Equation With Six Unknowns $x^2+y^2+z^4=u^3+v^4+(z+v)w^2$

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ABSTRACT

We obtain infinitely many non-zero integer sextuples (x,y,z,u,v,w) satisfying the biquadratic equation with six unknowns $x^2+y^2+z^4=u^3+v^4+(z+v)w^2$. Various interesting properties among the values of x, y, z, u, v and w are presented.

KEYWORDS: biquadratic equation with six unknowns,integral solutions.

MSC 2000 Mathematics subject classification: 11D25.

NOTATIONS:

$T_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$ -Polygonal number of rank n with size m

$P_n^m=\frac{1}{6}\left(n(n+1)((m-2)n+(5-n)\right)$ - Pyramidal number of rank n with size m

$SO_n=n(2n^2-1)$ -Stella octangular number ofrank n

$S_n=6n(n-1)+1$ -Star number of rank n

$OH_n=\frac{1}{3}\left(n(2n^2+1)\right)$ - Octahedral number of rank n

$J_n=\frac{1}{3}\left(2^n-(-1)^n\right)$ -Jacobsthal number of rank n

$j_n = 2^n + (-1)^n$ - Jacobsthal-Lucas number of rank n

INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems. In particular, biquadratic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context one may refer [6-9] for various problems on the biquadratic diophantine equations with four variables and [10] for five variables. However, often we come across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. It is towards this end this paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous equation with six unknowns given by $x^2 + y^2 + z^4 = u^3 + v^4 + (z+v)w^2$. A few relations among the solutions are presented.

Method of Analysis:

The diophantine equation representing the biquadratic equation with six unknowns under consideration is

$$x^2 + y^2 + z^4 = u^3 + v^4 + (z+v)w^2 \tag{1}$$

Different solutions pattern to (1) are presented below

Pattern1:

Introducing the transformations,

$$x = a(a^2 + b^2), y = b(a^2 + b^2), u = a^2 + b^2 \tag{2}$$

in (1), it simplifies to

$$z^4 - v^4 = (z+v)w^2 \tag{3}$$

Substituting

$$z = p+q, v = p-q, w = 2q\alpha, p \neq q \tag{4}$$

in (3), it reduces to the Pythagorean equation

$$(2p)^2 + (2q - \alpha^2)^2 = (\alpha^2)^2 \tag{5}$$

whose solution is

$$p = rs, q = r^2 \tag{6}$$

$$\alpha^2 = r^2 + s^2 \tag{7}$$

Now taking

$$r = 2ab, s = a^2 - b^2$$

we have

$$\alpha = a^2 + b^2 \tag{8}$$

Using (4), (6) and (8), we get the integral solution to (1) as

$$\left.\begin{aligned} x(a,b) &= a(a^2+b^2) \\ y(a,b) &= b(a^2+b^2) \\ u(a,b) &= (a^2+b^2) \\ v(a,b) &= 2ab(a^2-b^2) - 4a^2b^2 \\ w(a,b) &= 8a^2b^2(a^2+b^2) \\ z(a,b) &= 4a^2b^2 + 2ab(a^2-b^2) \end{aligned}\right\} \tag{9}$$

Properties:

1. $6(2x(a,1) + y(a,1) + u(a,1) - v(a,1) - S_a - 14T_{4,a} + 4T_{9,a})$ is a nasty number. 2. $8(x(a,1) + y(a,1) - 6P_a^3 + 6T_{6,a} + 4T_{3,a} - 2T_{4,a})$ is a cubical integer 3. $z(a,1) + v(a,1) = SO_a + 6P_a^3 + 2P_a^5 - 8T_{3,a} + T_{6,a} - 2T_{4,a}$ 4. $x(a,1) + y(a,1) + u(a,1) + v(a,1) + w(a,1) = 16P_{a^2}^5 - 4T_{4,a^2} + 6(OH_a) + T_{6,a} - 2T_{3,a} + T_{4,a}$
5. $u(2^{2a},1) + y(2^{2a-1},1) = j_{4a} + j_{4a-2}$

PatternI1:

Instead of (6) & (7) take the solution of (5) as

$$p = \frac{r^2 - s^2}{2}, q = \frac{r^2 + s^2 + 2rs}{2}, \alpha^2 = r^2 + s^2 \tag{10}$$

The values of r and s satisfying the system (10) are,

$$r = 4gh., s = 2(g^2 - h^2), g \neq h$$

Using (2) & (4), we get the integer solutions of (1) as

$$\left.\begin{aligned} x(a,b) &= 2gh(g^2+h^2)^2 \\ y(a,b) &= (g^2-h^2)(g^2+h^2)^2 \\ u(a,b) &= (g^2+h^2)^2 \\ v(a,b) &= -(g^2-h^2)(g^2-h^2+2gh) \\ w(a,b) &= 8(2gh+g^2-h^2)(g^2+h^2)^2 \\ z(a,b) &= 16g^2h^2 + 8gh(g^2-h^2) \end{aligned}\right\} \tag{11}$$

Properties:

$1.\ 2(8x(a,a)+y(a,a)+z(a,a)-u(a,a)+v(a,a)-w(a,a))$ is a nasty number.

$2.\ 8[u(a(a+1),1)-16T_{3,a}^{4}+2T_{4,a^2}+12P_a^4-8T_{4,a}+2T_{6,a}]$

$3.\ 16(v(a,1)+2u(a,1)-T_{4,a}^{2}+6p_a^4-18T_{3,a}-2T_{10,a}+8T_{4,a})$ is a biquadratic integer..

$4.\ x(2^{2a},1)+u(2^{a},1)=3(J_{10a+1}+J_{6a+2}+2J_{2a+1}+J_{4a})$

$5.\ x(a,a)+w(a,a)-432P_{a^2}^{3}+216T_{4,a^2}+288T_{3,a}\equiv 0 \bmod(12)$

PatternI1I:
Considering (5) & (8) and employing the method of factorization, define

$$(2p+i(2q-\alpha))=(a+ib)^4 \tag{12}$$

Equation real and imaginary parts of (12), we get

$$p(a,b)=\frac{1}{2}(a^4-6a^2b^2+b^4)$$

$$q(a,b)=2(a^3b-ab^3)+\frac{1}{2}(a^2+b^2)$$

Using (4) & (8), we get

$$\left.\begin{aligned} z(a,b)&=2ab(a^2-b^2)+(a^2-b^2)^2\\ v(a,b)&=-2ab(a^2-b^2)-4a^2b^2\\ w(a,b)&=(a^2+b^2)^3+4ab(a^4-b^4)\end{aligned}\right\} \tag{13}$$

The system of equations (2) & (13) gives the non-zero integer solutions of (1)

Properties:

$1.\ zu=uv+w$

$2.\ 8x^2y^2=u^6+wu^3-2zu^4$

$3.\ u^4(z+v)=u^6-8x^2y^2$

$4.\ u^6-wu^3=2vu^4+2(u^3-(x+y)^2)^2$

Remarks:

(1): Rewrite (5) as

$$(2p)^2+(2q-\alpha^2)^2=(\alpha^2)^2\times 1 \tag{14}$$

Write 1 as either

$$1=\frac{((m^2-n^2)+i2mn)((m^2-n^2)-i2mn)}{(m^2+n^2)^2} \qquad m\succ n \tag{15}$$

or

$$1 = \frac{(2mn + i(m^2 - n^2))(2mn - i(m^2 - n^2))}{(m^2 + n^2)^2} \qquad m \succ n \quad (16)$$

Following the analysis presented above, one may get non-zero distinct integer solutions of (1).In particular, when m=2, n=1 in (15) and performing a few algebra, the corresponding integer solutions of (1) are obtained as:

$$\left.\begin{aligned}
z &= 2(5)^3[(3a^4 + 3b^4 - 8a^2b^2) - (a^3b - ab^3)] \\
v &= -(5^3)[(3a^4 + 3b^4 + 2a^2b^2) + 14(a^3b - ab^3)] \\
w &= (5^5)(a^2 + b^2)[(9a^4 + 9b^4 - 14a^2b^2) + 12(a^3b - ab^3)] \\
x &= a(a^2 + b^2) \\
y &= b(a^2 + b^2) \\
z &= a^2 + b^2
\end{aligned}\right\} \quad (17)$$

(2): It is worth to mention here that, instead of (2), one may consider the values of x, y and u to be:

$$x = a^3 - 3ab^2$$
$$y = 3ab^2 - b^3$$
$$u = a^2 + b^2$$

and thus obtain a different choice of solutions to (1)

Conclusion:

One may search for other patterns of solutions and their corresponding properties.

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