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Pseudo-Spectral Sequences

On The Hyers-Ulam stability of
Nonlinear Differential Equation
$y'(t) = \alpha y(t) - \beta y^2(t)$

Habib Shakoory
Municipality Development planning department,
Tabriz, Iran
E-mail: HabibShakoory@yahoo.com

Abstract

This paper evaluates the differential equations of Urban population growth rate and discusses about exceptional points(solution points) from the stability point of view. It also offers sufficient conditions enabling us to approximate nonlinear differential equation with linear one. Although it is difficult and sometimes impossible to solve nonlinear differential equations finding qualitative and conceptual information however, is very important in this kind of equations.

AMS subject classification:
Keywords: Hyers-Ulam Stability, Differential equation, Approximation.

1. Introduction

In 1940 S.M. Ulam [1] proposed the famous stability problem of linear functions. In 1941 D.H. Hyers [1] considered the case of approximately additive functions $f : E \to E'$ where E and E' are Banach spaces and f satisfies the inequality $\|f(x + y) - f(x) - f(y)\| \leq \epsilon$ for all $x, y \in E$. It was shown that the limit $l(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive function satisfying $\|f(x)-L(x)\| \leq \epsilon$. In 1978 Th.M. Rassias generalized the result to the case of approximately additive function $f : E \to E'$ satisfying $\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in E$ and for some constants $\theta \geq 0$ and $0 \leq p < 1$. Since then, the stability problem has been widely investigated for different types of functional equations. T. Miura, S.-E. Takahashi, H. CHoda, [3] consider a differentiable map y from an open interval to a real Banach space of all bounded continuous real -valued functions on a topological space they show that y can be approximated by the solution to the differential equation $x'(t) = \lambda x(t)$, if $\|y'(t) - \lambda y(t)\|_\infty < \epsilon$ Holds. In [4] generalized this result to the case of the complex Banach space valued differential equation$y' = \lambda y$. Y. Li [2] is to prove stability in the sense of Hyers-Ulam of differential equation of second order $y'' = \lambda^2 y$ in this artical.

We say that equation $y'(t) = \alpha y(t) - \beta y^2(t)$ has the Hyers-Ulam stability if there exists a constant $K > 0$ With the following property:

- for every $t, \epsilon > 0$, $y \in C(I)$, if

$$|y'(t) - \alpha y(t) + \beta y^2(t)| \le \epsilon$$

- then there exists some $z \in C(I)$ satisfying

$$z' - \alpha z + \beta z^2 = 0$$

such that

$$|y - z| \le K\epsilon$$

We call such K a Hyers-Ulam stability constant for equation

$$y'(t) = \alpha y(t) - \beta y^2(t).$$

2. Main results

The differential equations of population growth rate are as follows:

$$y'(t) = \alpha y(t) - \beta y^2(t)$$

$$y(0) = y_0$$

In which $\beta$and α are two constants. Although it is mathematically possible to assign negative values to y_0, but it should be a positive number as far as physic rules concerns. If we neglect βy^2 term, the solution of the equation $y'(t) = \alpha y(t) - \beta y^2(t)$ with primary condition $y(0) = y_0$ would be $y = y_0 e^{\alpha t}$.

In which, y increases exponentially. If the term be kept linear, a different result will be obtained. Two $y = \alpha/\beta = \varphi_2(t)$and $y = \varphi_1(t) = 0$ functions are the solutions of the equation $y'(t) = \alpha y(t) - \beta y^2(t)$. Now if we assume that $y_0 \neq 0$ and $y_0 \neq \alpha/\beta$ the two functions would not be the solutions of the equation with the mentioned primary condition. In this case we can write:

$$\partial y / y(\alpha + \beta y) = \partial t$$

or:

$$(1/y + \beta/(\alpha - \beta y))\partial y = \partial t$$

With the integration of the parties we have:

$$\log|y| - \log|\alpha - \beta y| = \alpha t + k$$

Constant k should be selected so that the condition of $y(0) = y_0$ be met. Since $\alpha - \beta y$ has different signs within $0 < y < \alpha/\beta$ and $y > \alpha/\beta$ ranges, the two ranges should be considered separately. The final result could be summarized as follows:

$$y = \varphi(t) = \alpha/(\beta + ((\alpha - \beta y_0)/(y_0))e^{-\alpha t}, y_0 \neq 0$$

Although the solution of $y_0 = 0$ is not included in this relation but the solution of $y_0 = \alpha/\beta$ is the specific state of it. According to our assumption $\epsilon > 0$ therefore, when $t \to \infty$, $e^{-\alpha t} \to 0$ and $y \to \alpha/\beta$.

Therefore, each solution of the equation $y'(t) = \alpha y(t) - \beta y^2(t)$ with every probable primary value y_0 would approach $y = \varphi_t = \beta/\alpha$ when $t \to \infty$. This is that qualitative concept we consider in such problems. In this equation we concluded the limit state of it following finding its solution but this case is more important in sophisticated problems where the limit state is obtainable even without finding its solution. In order to explain this more, we draw the graph of $\partial y/\partial t$ as a function of y.

Here, the solution of $y = \varphi_1(t)$ is studied in detail. To show our general purpose of this study, we assume that the differential equation $y'(t) = \alpha y(t) - \beta y^2(t)$ is the graph of a physical device which is shown as rectangular 4.

If $y_0 = 0$ is considered as the feed of this device, its efficiency would be $y = 0$. Suppose that due to an error, instead of $y_0 = 0$ a small positive value i.e. $y_0 > 0$ is fed to the device. A question which will be arose here is that whether or not the efficiency of the device approaches y=0. The answer of this question is no and y would approach β/α instead of zero. This case is named as an unstable solution of $y = \varphi_1(t) = 0$. This means that even a small change in the primary condition i.e. device feed, would result in a great change of device response i.e. its efficiency. Now we discuss the solution of $y = \varphi_2(t) = \alpha/\beta$. In the case where the feed of the device is $y_0 = \alpha/\beta$ its efficiency would be $y = \alpha/\beta$ as well. In the event of small error in feed, the efficiency would exponentially approaches β/α. This case is called an asymptotically stable solution of the equation $y'(t) = \alpha y(t) - \beta y^2(t)$.

Now we consider the differential equation $y'(t) = \alpha y(t) - \beta y^2(t)$ and the primary condition $y(0) = y_0$ and $\alpha < 0$ and $\beta < 0$ and without a definite solution process we evaluate that what solutions could be obtained in terms of y from $\partial y/\partial t$ relation.

Since for $0 < y < \alpha/\beta$, $\partial y/\partial t$ is negative therefore if $0 < y_0 < \beta/\alpha$ then y decreases and approaches $y = \varphi_1(t) = 0$.

On the other hand if $y > \alpha/\beta$, $\partial y/\partial t$ would be positive. Therefore, if $y_0 > \beta/\alpha$, y increases and approaches infinity as t increases.

From this discussion we can conclude that any small positive change in the primary condition of $y_0 = \alpha/\beta$ can result in the increase of y up to infinity and in contrast, any small negative change can lead to its decrease up to zero. Therefore, the solution of $y = \varphi_2(t) = \alpha/\beta$ is instable in this condition.

On the other hand in this example the solution of $y = \varphi_1(t) = 0$is an asymptotically stable solution because any small change in the primary condition of y_0 only changes the value of y at initial times which approaches the primary solution of $\varphi_1(t)$ as t increases. It should be mentioned that this stability is valid only for small changes in the primary condition i.e. for values smaller than α/β and in the event of any fault in feed value in which the error is higher than this range, the efficiency will approach infinity instead of $y = \varphi_1(t)$ solution.

Table 6 shows the curves of different solutions for the primary values of $y_0, \alpha < 0$ and $\beta < 0$. Finally, we discuss the linear conversion of the equation $y'(t) = \alpha y(t) - \beta y^2(t)$ in which we neglect βy^2 term against αy.

In the first state i.e. when $\alpha > 0$and $\beta > 0$ if $y_0 = y(0)$ is small enough, for short term times linear equations would be a relatively good approximation for nonlinear one while for long term times it is not true to neglect βy^2 term against αy because although it is possible that the primary value of y is small enough but the value of y increases exponentially and αy^2 would be comparable with αy. In the second state i.e. when $\alpha < 0$ and $\beta < 0$ the solution of linear equation of $y_0 e^{\alpha t}$ approaches zero for every value of y_0. Also, the solution of nonlinear equation of $y_0 < \alpha/\beta$ approaches zero as well. In the first state the solution of linear equation is not a good approximation for nonlinear equation (at least for larger t(s)) while in the second state for every small value of y_0 i.e. $y_0 < \alpha/\beta$, linear equation is considered as a good approximation for nonlinear one.

Now , the main result of this work is given in the following theorem.

Theorem 2.1. If a continuously differentiable function $y : I \to R$ satisfies the differential inequality

$$|y'(t) - \alpha y(t) + \beta y^2(t)| \leq \varepsilon$$

for all $t \in I$ and for some $\epsilon > 0$, then there exists a solution $v : I \to R$ of the equation :

$$|v'(t) - \alpha v(t) + \beta v^2(t)| \leq \varepsilon$$

,such that

$$|y(x) - v(x)| \leq K\epsilon$$

Where $K > 0$ is a constant.

Proof. According to the above description, We can use linear equation $(y'(t) = \alpha y(t))$ instead of the nonlinear one$(y'(t) = \alpha y(t) - \beta y^2(t))$, That is stability was proven by S.-E. Takahasi, T. Miura, S. Miyajima in [4] and T. Miura, S.-E. Takahasi, H. CHoda in [3], Which it is not need for show. ■

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