Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 8, Number 5 (2012), pp. 509–515

http://www.ripublication.com/gjpam.htm

Behaviour of Faber series at an analytic point of the Boundary

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Abstract

We obtain the Fatou type theorem which reflects the behaviour of the Faber series on the its equipotential line of convergence.

AMS subject classification:
Keywords:

1. Introduction

Let E be a permissible continuum [1], By $F(z)$ denote the Faber polynomials of E. The Faber polynomial is a special polynomial, it depends on the set E. For example, if E is a closed disk, then its Faber polynomial is just the common power function.

Suppose that a conformal bijection $W = \varphi(z)$ maps the exterior of E onto $|W| > 1$ and satisfies the following two conditions

$$\varphi(\infty) = \infty, \lim_{z \to \infty} \frac{\varphi(z)}{z} = c(0 < c < \infty)$$

By Γ_λ denote the equipotential lines $|\varphi(z)| = \lambda < \lambda$ of E. It is well known that for a Faber series

$$\sum_{0}^{\infty} a_n F_n \tag{1}$$

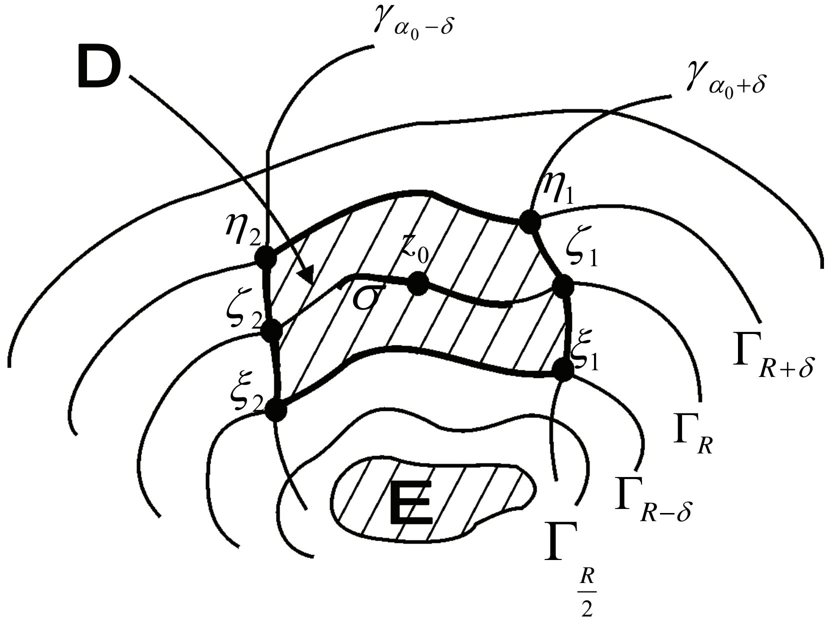

Figure 1: Curvilinear quardrilateral D.

Its curve of convergence is a equipotential line of E[1].

It is also well known that in the complex plane for realizing the approximation by polynomials, the Faber polynomial is a useful approximation tool. So the research of Faber polynomial is a valuable topic. In the literature many authors have studied Faber expansion, the approximation by Faber polynomial, the overconvergence of Faber series and so on by now, eg. see [2]–[7]. In this paper we shall discuss the behaviour of Faber series on its equipotential line of convergence under the mixed conditions.

2. Main result

In this paper we shall give the following main result.

2.1. Theorem (Fatou type theorem)

Let an equipotential line $\Gamma_\lambda : |\varphi(z)| = R$ $(R. > 1)$ be the curve of convergence of the Faber series (1), and let

$$f(z) = \sum_{0}^{\infty} a_n F_n \text{ (z is inside of } \Gamma_R).$$

Suppose that the coefficients are $a_n = o(R^{-n})(n \to \infty)$ and $f(z)$is regular at a point $z_0 \in \Gamma_\lambda$. Then series (1) is convergent uniformly in an arc on Γ_λ with the 'center' z_0 $(z_0 \in \Gamma_R)$ (see the Fig. 1).

Aspecial feature in theTheorem is, when E is the closed disk $|z - z_0| \leq 1$ its equipotential lines Γ_R are the circle $|z - z_0| = R$ and its Faber polynomials are $(z - z_0)^n$. So using Theorem, we can get the known Fatou theorem of power series [8, pp 218] immediately.

When E is closed interval $[-1, 1]$, the equipotential lines $\Gamma_R(R > 1)$ are the ellipse $|z-1| + |z+1| = R + \dfrac{1}{R}$ and its Faber polynomials

$$F_0(z) = 1, F_n(z) = 2\cos n\ arc \cos z$$

are the known Chebyshev polynomials $T_n(z)$[1]. so we can get the following Fatou type theorem for Chebyshev series.

Corollary 1

By $T_n(z)$ denote the Chebyshev polynomials. Let the ellipse $\Gamma_R : |z-1| + |z+1| = R + \dfrac{1}{R}(R > 1)$ be the curve of convergence of the Chebyshev series and

$$f(z) = \sum_0^\infty a_n F_n \ (z \text{ is inside of } \Gamma_R).$$

Suppose that the coefficients $a_n = o(R^{-n})(n \to \infty)$, and $f(z)$ is regular at a point $z_0 \in \Gamma_\lambda$. Then the Chebyshev series is convergent uniformly in an arc on Γ_R with the 'center' $z_0(\ z_0 \in \Gamma_R)$.

For Laguarre series which is not a Faber series, the corresponding Fatou type theorem has been obtained in [9].

Proof of Theorem 1

Write $\alpha_0 = Arg\ \varphi(z_0)$ Take two curves $\gamma_{\alpha_0 + \delta}$ and $\gamma_{\alpha_0 - \delta}(\delta > 0)$ whose equations are $Arg\ \varphi(z) = \alpha_0 + \delta$ and $Arg\ \varphi(z) = \alpha_0 - \delta$ respectively.

Let the curves $\gamma_{\alpha_0 + \delta}$ and $\gamma_{\alpha_0 - \delta}$ intersect at points $\xi_i, \eta_i (i = 1, 2)$ with the equipotential lines $\Gamma_{R+\delta}$ and $\Gamma_{R-\delta}$, and intersect at $\zeta_i(i = 1, 2)$with Γ_R (see the Fig. 1).

Because $f(z)$ is regular at z_0, we can choose a fixed $\delta > 0$ so small that is regular in a closed curvilinear quardrilateral $\overline{D}$ with the vertexes $\xi_i, \eta_i(i = 1, 2)$ (see the Fig. 1).

Let

$$P_n(z) = \frac{R^n(\varphi(z) - \varphi(\zeta_1))(\varphi(z) - \varphi(\zeta_2))}{F_n(z)} \times \left(f(z) - \sum_0^n a_i F_i(z) \right) \tag{2}$$

Below we shall estimate $|P_n(z)|$ on the boundary ∂D of the above curvilinear quardrilateral D.

(i) First we estimate $|P_n(z)|$ on the curve $\widehat{\xi_1 \zeta_1} \subset (\gamma_{\alpha_0 + \delta})$.

It is clear that for the end point ζ_1, $P_n(\zeta_1) = 0$.

By assumption,

$$f(z) - \sum_{0}^{n} a_j F_j(z) = \sum_{n+1}^{\infty} a_i F_i(z), \quad \text{for } z \in \widehat{\xi_1 \zeta_1} - \{\zeta_1\} \tag{3}$$

However, for Faber Polynomials, we know that [1] for $R > 1$ there exists an $N(R)$such that when $n > N(R)$,

$$\frac{1}{2}|\varphi(z)|^n \leq |F_n(z)| \leq \frac{3}{2}|\varphi(z)|^n \text{ for } z \in \Gamma_{\frac{R}{2}} \text{ or inside } \Gamma_{\frac{R}{2}} \tag{4}$$

Again by the known condition $a_n = o(R^{-n})(n \to \infty)$, we can get that for any $\epsilon > 0$ there exists an $N_1 > N(R)$ such that

$$|a_n| < \epsilon R^{-n} \; (n > N_1) \tag{5}$$

So combining this with (4) and noticing that

$$|\varphi(z)| < R \quad \text{for } z \in \widehat{\xi_1 \zeta_1} - \{\zeta_1\} \tag{6}$$

We obtain from (3) that

$$|f(z) - \sum_{0}^{n} a_i F_i(z)| \leq \frac{3}{2}\epsilon \sum_{n+1}^{\infty} \left(\frac{|\varphi(z)|}{R}\right)^i \leq \frac{3}{2}\epsilon \frac{|\varphi(z)|^n}{R^{n-1}(R - |\varphi(z)|)} \quad ,(n > N_1)$$

From this and (2), (4), we obtain

$$|P_n(z)| = \frac{\epsilon R|(\varphi(z) - \varphi(\zeta_1))(\varphi(z) - \varphi(\zeta_2))|}{R - |\varphi(z)|}, n > N_1 \text{ for } z \in \widehat{\xi_1 \zeta_1} - \{\zeta_1\} \tag{7}$$

By $\varphi(z) = |\varphi(z)|e^{i(\alpha_0+\delta)}$,for $z \in \widehat{\xi_1 \zeta_1}$, and

$$\varphi(\zeta_1) = \mathrm{Re}^{i(\alpha_0+\delta)}, \quad \varphi(\zeta_2) = \mathrm{Re}^{i(\alpha_0-\delta)} \tag{8}$$

We have by (6),

$$|\varphi(z) - \varphi(\zeta_1)| = R - |\varphi(z)|, \qquad |\varphi(z) - \varphi(\zeta_2)| = 2R \quad \text{for } z \in \widehat{\xi_1 \zeta_1} - \{\zeta_1\}$$

Again by (7), we have
$|P_n(z)| \leq K\epsilon, \quad n > N_1, \quad \text{for } z \in \widehat{\xi_1 \zeta_1} - \{\zeta_1\}$.
here and later K are different positive constants which are independent of n, z.
(ii) Next we estimate $|P_n(z)|$ on the curve $(\widehat{\zeta_1 \eta_1} - \{\zeta_1\}) \subset \gamma_{\alpha_0 + \delta}$
Because $f(z)$ is bounded in $\overline{D}$,

$$|f(z) - \sum_{0}^{n} a_j F_j(z)| \leq K + \sum_{N_1+1}^{n} |a_i F_i(z)|, \; n > N_1 \quad \text{for } z \in \overline{D} \tag{9}$$

In view of

$$R < |\varphi(z)| < R + \delta \qquad \text{for } \widehat{\zeta_1 \eta_1} - \{\zeta_1 \tag{10}$$

by (4) and (5), we obtain

$$\sum_{N_1+1}^{n} |a_i F_i(z)| \leq \frac{3}{2}\epsilon \frac{(R+\delta)|\varphi(z)|^n}{R^n(|\varphi(z)|-R)}.$$

Combining this with (4) and (9), it follows that from (2) that

$$|P_n(z)| \leq \frac{2R^n|(\varphi(z)-\varphi(\zeta_1))(\varphi(z)-\varphi(\zeta_2))|}{|\varphi(z)|} \times \left\{K + \frac{3}{2}\epsilon \frac{(R+\delta)|\varphi(z)|^n}{R^n(|\varphi(z)|-R)}\}, n > N_1, z \in \widehat{\zeta_1\eta_1} - \{\zeta_1\right\} \quad (11)$$

But by (8) and (10), we have for $z \in \widehat{\zeta_1\eta_1} - \{\zeta_1\}$

$$|\varphi(z)-\varphi(\zeta_2)| = 2R+\delta, \qquad |\varphi(z)-\varphi(\zeta_1)| = |\varphi(z)| - R$$

and

$$\frac{|\varphi(z) - \varphi(\zeta_1)|}{|\varphi(z)|^n} \leq \frac{|\varphi(z)| - R}{|\varphi(z)|^n - R^n} = \frac{1}{|\varphi(z)|^{n-1} + |\varphi(z)|^{n-2}R + \cdots + R^{n-1}} \leq \frac{1}{nR^{n-1}},$$

So by (11),

$$|P_n(z)| \leq K\frac{2R^n|(\varphi(z)-\varphi(\zeta_1))(\varphi(z)-\varphi(\zeta_2))|}{|\varphi(z)|^n} + 3\epsilon(R+\delta)(2R+\delta) \leq \frac{2R(2R+\delta)K}{n} + 3\epsilon(R+\delta)(2R+\delta) \quad \text{for } z \in \widehat{\zeta_1\eta_1} - \{\zeta_1\}$$

Hence there is an $N_2 > N_1$ such that for $n > N_2$.

$$|P_n(z)| \leq K\epsilon, \qquad \text{for } z \in \widehat{\zeta_1\eta_1} - \{\zeta_1\}$$

(iii) Finally, we estimate $|P_n(z)|$ on the curve $\widehat{\eta_1\eta_2} \subset (\Gamma_{R+\delta})$.

In view of for

$$|\varphi(z)| = R+\delta, \qquad \text{for } z \in \widehat{\eta_1\eta_2} \quad (12)$$

We obtain from (4) and (5) that

$$\sum_{N_1+1}^{n} |a_i F_i(z)| \leq \frac{3R\epsilon}{2\delta}\left(\frac{R+\delta}{R}\right)^{n+1}, \qquad \text{for } z \in \widehat{\eta_1\eta_2}$$

Combining this with (4), (9) and (12), we obtain from (2) that

$$|P_n(z)| \leq \frac{2R^n|\,(\varphi(z)-\varphi(\zeta_1))(\varphi(z)-\varphi(\zeta_2))\,|}{(R\ +\ \delta)^n} \times \left\{K + \frac{3R\epsilon}{2\delta}\left(\frac{R\ +\ \delta}{R}\right)^{n+1}\right\}, n > N_1 \quad \text{for } z \in \widehat{\eta_1\eta_2}$$

However, by (8) and (12),

$$|P_n(z)| \leq K\left\{\left(\frac{R\ +\ \delta}{R}\right)^n + \epsilon\right\}, n > N_1 \quad \text{for } z \in \widehat{\eta_1\eta_2}$$

So there exists N_3N_2 such that $\left(\frac{R\ +\ \delta}{R}\right)^n < \epsilon$, $(n > N_3)$.
Furthermore

$$|P_n(z)| \leq K\epsilon\ , \qquad n > N_3 \quad \text{for } z \in \widehat{\eta_1\eta_2}.$$

So combining **(i)**, **(ii)** and **(iii)** we obtain finally that

$$|P_n(z)| \leq K\epsilon\ , \qquad n > N_3 \quad \text{for } z \in \widehat{\eta_1\eta_2} \cup \widehat{\xi_1\eta_1}$$

Similarly, we can also obtain that

$$|P_n(z)| \leq K\epsilon\ , \qquad n > N_3^* \quad \text{for } z \in \widehat{\xi_1\xi_2} \cup \widehat{\xi_2\eta_2}\ .$$

Consequently we have by the maximum modulus principle that

$$|P_n(z)| \leq K\epsilon\ , \qquad \text{for } z \in \overline{D}, \quad n > N = \max\{N_3, N_3^*\}.$$

Now on Γ_R, take a neighbourhood σ:$|Arg\varphi(z)| \leq \alpha_0 + \frac{\delta}{2}$, $\quad z \in \Gamma_R$ with the center z_0, where it is clear that $\sigma \subset \overline{D}$ In this neigbourhood.
We see that $\varphi(z) = \mathrm{Re}^{i(\alpha_0+t)}$, so by (8),we obtain

$$|\varphi(z) - \varphi(\zeta 1)| = R|e^{i(\alpha_0+t)} - e^{i(\alpha_0+\delta)}| \geq 2R\sin\frac{\delta}{2} \quad \text{for } z \in \sigma$$

and

$$|\varphi(z) - \varphi(\zeta_2)| \geq 2R\sin\frac{\delta}{2} \quad \text{for } z \in \sigma$$

Thus by (2), (4) and (13), we obtain:

$$\left|f(z) - \sum_0^n a_i F_i(z)\right| \leq \frac{K\epsilon}{8R^2\sin^2\frac{\delta}{2}}, \qquad \text{for } z \in \sigma, n > N$$

i.e. $\sum_0^n a_i F_i(z)$ is convergent uniformly in σ.
The proof of Theorem is completed. ■

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