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Entropy Maximization through Dynamic Programming

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Abstract

Dynamic programming is an optimization technique based on Bellman's principle of optimality. In this paper we have maximized the Kapur's entropy by using dynamic programming under constraints $\sum_{i=1}^{n} p_i = c$, $p_i \geq 0$.

Keyword: Entropy, optimization, optimality, Dynamic Programming.

Introduction

Bellman's principle of optimality is "An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision", and the dynamic programming is based on it.

Kapur (1968b,1972) used dynamic programming for maximizing measure of entropy subject to given constraints and for optimal sub division of out-comes for getting maximum gain in information subject to a given budget. Kapur[1] maximized Shannon's[3] entropy

$$-\sum_{i=1}^{n} p_i \ln p_i \tag{1}$$

subject to the constraints

$$\sum_{i=1}^{n} p_i = c, p_i \geq 0, i = 1, 2, ..., n \tag{2}$$

by dynamic programming.

In this paper we have maximized Kapur's measure of entropy subject to (2) by dynamic programming.

Entropy Maximization

Let the probabilities of n possible outcomes $x_1, x_2, ..., x_n$ of an experiment be respectively $p_1, p_2, ..., p_n$. Kapur [2] suggested the following measure

$$H_n(P) = \sum_{i=1}^{n} \left[\ln(1+ap_i) - p_i \ln(1+a) \right], \ a > 0 \tag{3}$$

For the application of this principle, we consider maximization of

$$H_n(P) = \sum_{i=1}^{n} \ln(1+ap_i) - \sum_{i=1}^{n} p_i \ln(1+a)$$

subject to the constraints

$$\sum_{i=1}^{n} p_i = c, \ p_i \ge 0, i = 1, 2, ..., n. \tag{4}$$

Let the maximum value be $H_n(c)$, then obviously

$$H_1(c) = \ln(1+ac) - c\ln(1+a) \tag{5}$$

If we choose p_1 arbitrarily between 0 & c, we have to maximize

$$-\sum_{j=2}^{n} \ln(1+ap_j) - \sum_{j=2}^{n} p_j \ln(1+a)$$

subject to the constraints

$$\sum_{j=2}^{n} p_j = c - p_1, \ p_i \ge 0, i = 1, 2, ..., n.$$

This maximum value will be $H_{n-1}(c-p_1)$. The principle of optimality then gives the recurrence relation

$$H_n(c) = \max_{0 \le p_1 \le c} \left[\left(\ln(1+ap_1) - p_1 \ln(1+a) \right) + H_{n-1}(c-p_1) \right] \tag{6}$$

Putting $n = 2$, we get on using (5)

$$H_2(c) = \max_{0 \le p_1 \le c} \left[\left(\ln(1+ap_1) - p_1 \ln(1+a) \right) + H_1(c-p_1) \right]$$

$$= \max_{0 \le p_1 \le c} \left[\left(\ln(1+ap_1) - p_1 \ln(1+a) \right) + \ln(1+ac-ap_1) - (c-p_1)\ln(1+a) \right]$$

$$= \ln\left(1+\frac{ac}{2}\right)+\ln\left(1+\frac{ac}{2}\right)-c\ln(1+a)$$
$$= 2\ln\left(\frac{2+ac}{2}\right)-c\ln(1+a) \qquad (7)$$

and maximum value arises when $p_1 = p_2 = \frac{c}{2}$. Similarly putting $n=3$ in (6), we get, on using (7)

$$H_3(c) = \max_{0\le p_1\le c}\left[\left(\ln(1+ap_1)-p_1\ln(1+a)\right)+H_2(c-p_1)\right]$$
$$= \max_{0\le p_1\le c}\left[\left(\ln(1+ap_1)-p_1\ln(1+a)\right)+2\ln\left(\frac{2+ac-ap_1}{2}\right)-(c-p_1)\ln(1+a)\right]$$
$$= \ln\left(\frac{3+ac}{3}\right)+2\ln\left(\frac{3+ac}{3}\right)-c\ln(1+a)$$
$$= 3\ln\left(\frac{3+ac}{3}\right)-c\ln(1+a) \qquad (8)$$

and the maximum value arises when $p_1 = p_2 = p_3 = \frac{c}{3}$.

It thus appears that

$$H_n(c) = n\ln\left(\frac{n+ac}{n}\right)-c\ln(1+a) \qquad (9)$$

and this arises when

$$p_1 = p_2 = ... = p_n = \frac{c}{n} \qquad (10)$$

Assuming it to be true for a specific value of n, equation (8) gives

$$H_{n+1}(c) = \max_{0\le p_1\le c}\left[\left(\ln(1+ap_1)-p_1\ln(1+a)\right)+H_n(c-p_1)\right]$$
$$= \max_{0\le p_1\le c}\left[\ln(1+ap_1)-p_1\ln(1+a)+n\ln\left(\frac{n+ac-ap_1}{n}\right)-c\ln(1+a)\right]$$
$$= (n+1)\ln\left(\frac{n+1+ac}{n+1}\right)-c\ln(1+a) \qquad (11)$$

Thus from (5), (9) and (10) and the principle of mathematical induction, the result (9) and (10) are true for all value of n and c. Putting $c=1$, we get the results that $\sum_{i=1}^{n}\ln(1+ap_i)-\sum_{i=1}^{n}p_i\ln(1+a)$ is maximum subject to $\sum_{i=1}^{n}p_i=1$ when

$$p_1 = p_2 = \ldots\ldots = p_n = \frac{1}{n}$$

and the maximum value is

$$n\ln\left(\frac{n+a}{n}\right) - \ln(1+a)$$

which is Kapur's[2] result.

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