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Fibrewise ℬ-Property

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Abstract

In [2] and [3], Buhagiar defined paracompact mappings, metacompact mappings, subparacompact mappings, submetacompact mappings and collectionwise normal mappings which are the fibrewise topological analogues of paracompact, metacompact, subparacompact, submetacompact and collectionwise normal spaces respectively. Also in [7], Al-Zoubi and Hdeib defined countably paracompact mappings which are the fibrewise topological analogues of countably paracompact spaces.

In this paper we define fibrewise ℬ-property on spaces. Several characterizations of fibrewise ℬ-property on spaces are proved. Several properties of fibrewise ℬ-property on spaces are studied.

Keywords: fibrewise ℬ-property, fibrewise paracompact, fibrewise, fibrewise Lindelof, fibrewise countably compact.

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Introduction

The fibrewise viewpoint is standard in the theory of fibre bundles, and some of the ideas of fibrewise topology originated in work of the ideas of that theory. In fibrewise we work over a topological base space B. When B is a point space, the theory reduced to that of ordinary topology. Most of the results obtained in this field can be found in [2], [3] and [6].

Preliminaries

Unless otherwise stated, B is a fixed topological space, $\overline{U}^{X_W}$ denotes the closure of U in X_W and nbhd denotes a neighborhood of a point in a topological space.

Definition 1.1 [10]: Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a collection of subsets of space X. Then $\mathcal{U}$ is called *monotone increasing* if $U_\alpha \subseteq U_\beta$ for any $\alpha < \beta$. $\mathcal{U}$ is called *locally finite* if for every $x \in X$ there is nbhd O of X such that the cardinality of $\{U \in \mathcal{U} \mid U \cap O \neq \phi\}$ is finite.

Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of a space X. Then we say that $\mathcal{V}$ is *cushioned* in $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U(V) \in \mathcal{U}$ such that for every subcollection $\mathcal{V}'$ of $\mathcal{V}$, $\overline{\cup\{V \mid V \in V'\}} \subseteq \cup\{U(V) \mid V \in \mathcal{V}'\}$.A collection ζ of subsets of space X is said to be *finitely closure - preserving* in X if for any subcollection ζ' of ζ and any point $x \in X$ there exists a nbhd O of x and a finite subcollection ζ'' of ζ' such that $O \cap (\cup \zeta') \subseteq \overline{U\zeta''}$.

Definition 1.1 [6]: let B be any set .Then a fibrewise over B consists of a set X together with a function $P : X \to B$, called the projection., where B is called a base set .

For each point b of B the fibre over is the subset $X_b = p^{-1}(b)$ *of* X .Also for each subset B' of B we regard $X_{B'} = p^{-1}(B')$ as fibrewise set over B' .

Definition 1.2 [6]: If X and Y are fibrewise sets over B with projections p and q respectively, a function $\varphi : X \to Y$ is said to be fibrewise if $q\varphi = p$. In other words $\varphi(X_b) \subseteq Y_b$ for each $b \in B$.

Definition 1.3[6]: Let $\{X_r\}$ be an indexed family of fibrewise sets over B. Then the fibrewise product $\prod_B X_r$ is defined as a fibrewise set over B, and comes equipped with the family of projections $\pi_r : \prod_B X_r \to X_r$

Specifically the fibrewise product is defined as the subset of the ordinary product $\prod_B X_r$ in which the fibrewise are the corresponding fibres of the factors X_r.

Definition 1.4 [6]: Let B be a topological space. Then a fibrewise topology on a fibrewise set X over B is any topology on X for which the projection p is continuous.

A fibrewise topological space over B is defined to be a fibrewise set over B with fibrewise topology.

If the projection p is closed then the fibrewise topology on X is called fibrewise closed

Proposition 1.5 [6]: Let X be fibrewise topological over B. Then X is fibrewise closed if and only if for each fibre X_b of X and each nbhd U of X_b in X there exists a nbhd W of b in B such that $X_W \subseteq U$.

Definition 1.6[6]:The fibrewise topological space X over B is called fibrewise T_1 if for every $x, y \in X_b$, $b \in B$,there exists a nbhd U of x which does not contain y and a nbhd V of y which does not contain x.

Definition 1.7[6] :The fibrewise topological space X over B is called fibrewise T_2 if whenever $x, y \in X_b$, $b \in B$, there exists disjoint nbhds of x and y in X.

Definition 1.8[6] :The fibrewise topological space X over B is called fibrewise regular if for each $x \in X_b$ and for each nbhd V of x in X, there exists a nbhd W of b in B and a nbhd U of x in X_W such that ……….

A fibrewise T_1 and fibrewise regular space is called fibrewise T_3-space.

Definition 1.9 [6]: Let X be fibrewise topological over B. Then X is called fibrewise compact if the projection P is perfect .i.e. X is fibrewise compact if and only if X is fibrewise closed and every fibre of X is compact.

Definition 1.10 [2]: Let X be fibrewise topological over B. For $b \in B$, a collection $\mathcal{U}$ of subsets of X is said to be b-locally finite if for every $x \in X_b$, there exists a nbhd V_x of x such that V_x meets only finitely many members of $\mathcal{U}$.

If the collection $\{U_\alpha \mid \alpha \in \Delta\}$ is b-locally finite open (in X) collection then $\mathcal{U}$ is locally finite in $\bigcup \{V_x \mid x \in X_b\}$.

Theorem 1.11 [2]: Let X be fibrewise topological over B. Then the following are equivalent:

1. X is fibrewise compact.
2. The fibrewise projection $\pi_Y : X \times_B Y \to Y$ is closed.

Definition 1.12 [2]: Let X be a fibrewise topological space over B. Then X is called fibrewise paracopmact if for every $b \in B$ and every open (in X) over $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ of X_b there is a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and $\{U_\alpha \cap X_b \mid \alpha \in \Delta\}$ has an open (in X) locally finite refinement in X_w.

Definition 1.13 [2]: Let X be fibrewise topological over B. Then X is called fibrewise Lindelof if X is fibrewise closed and for every $b \in B$, every open cover of of X_b has a countable subcover.

Theorem 1.14 [2]:a fibrewise topological space over B is fibrewise Lindelof if and only if for every $b \in B$ and every open cover $\mathcal{U}$ of X_b there exists a nbhd W of b in B such that X_w is covered by a countable subfamily from $\mathcal{U}$.

Theorem 1.15 [2]: If X is fibrewise Lindelof over B then X is fibrewise paracompact over B.

Definition1.16 [7]: Let X be fibrewise topological over B. Then X is called fibrewise countably paracompact if for every $b \in B$ and every countable open cover $\mathcal{U} = \{U_n \mid n \in N\}$ of X_b there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and $\{U_n \cap X_w \mid U_n \in \mathcal{U}\}$ has a locally finite open refinement in X_w.

Theorem 1.17 [7]: A fibrewise topological space X over B is fibrewise countably paracompact if and only if for every $b \in B$ and every monotone increasing open (in X) cover $\mathcal{U} = \{U_n \mid n \in N\}$ of X_b, there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and there exists a countable open (in X) cover $\mathcal{V} = \{V_n \mid n \in N\}$ of X_w such that $\overline{V_n}^{X_w} \subseteq U_n \cap X_w$.

Theorem 1.18[1]: Let X be fibrewise topological over B. then the following are equivalent:

X is fibrewise paracompact.

For every $b \in B$ and every monotone increasing open (in X) cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ of X_b, there are a nbhd W of b in B and monotone increasing and finitely closure-preserving open (in X) cover $\{V_\alpha \mid \alpha \in \Delta\}$ of X_w in X_w such that X_w is covered by $\mathcal{U}$ and $\overline{V_\alpha}^{X_w} \subseteq U_\alpha \cap X_w$ for every $\alpha \in \Delta$.

Fibrewise $\mathcal{B}$-Property

In this section we introduce the definition of fibrewise $\mathcal{B}$-Property on topological spaces or spaces that has $\mathcal{B}$-Property fibrewisely and give some characterizations of such spaces.

Definition 2.1: A fibrewise topological space X over B is said to have $\mathcal{B}$-Property fibrewisely over B if for every $b \in B$ and every monotone increasing open (in X) cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ of X_b there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and there exists a monotone increasing open cover $\mathcal{V} = \{V_\alpha \mid \alpha \in \Delta\}$ of X_w in X_w such that $\overline{V_\alpha}^{X_w} \subseteq U_\alpha \cap X_w$.

It is clear that:

Theorem 2.2: Every fibrewise paracompact space over B has $\mathcal{B}$-Property fibrewisely.

The converse of the last theorem is not true (see [10]).

Also from theorem 1.17 we have:

Theorem 2.3: Every fibrewise topological space that has $\mathcal{B}$-Property fibrewisely over B is fibrewise countably paracompact over B.

Theorem 2.4: Let X be a fibrewise topological space over B. Then the following are equivalent:

X has $\mathfrak{B}$-Property fibrewisely.

For every $b \in B$ and every monotone increasing open cover $\mathcal{U}$ of X_b, there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and $\{U_\alpha \cap X_w \mid \alpha \in \Delta\}$ has a cushioned open refinement in X_w. Then there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and $\{U_\alpha \cap X_w \mid \alpha \in \Delta\}$has a

For every $b \in B$ and every monotone increasing open cover $\mathcal{U}$ of X_b, there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and $\{U \cap X_w \mid \alpha \in \Delta\}$has a σ-cushioned open refinement in X_w.

Proof:

(1) $\to$ (2) Clear.

(2) $\to$ (3) Clear.

(3)$\to$(1) Let $b \in B$ and $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a monotone increasing open (in X) cover of X_b. Then there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and $\{U_\alpha \cap X_w \mid \alpha \in \Delta\}$ has a σ-cushioned open refinement $\cup\{\mathcal{V}_n \mid n \in N\}$, where each $\mathcal{V}_n$ is cushioned in $\mathcal{U}$. Let $\mathcal{V}_n = \{V_{n\alpha} \mid \alpha \in \Delta\}$, where $\overline{\cup\{V_{n\alpha} | \alpha \in \Delta'\}}^{X_w} \subseteq \cup\{U_\alpha | \alpha \in \Delta'\}$ for any subset Δ' of Δ and any $n \in \mathbb{N}$. Let $V_n = \cup\{V_{n\alpha} | \alpha \in \Delta\}$. Then $\{V_n | n \in N\}$ is a countable open (in X) cover of X_w. Since ,by theorem 1.17 , X is fibrewise countably paracompact, there exists a nbhd $W' \subseteq W$ of b in B such that $X_{W'}$ is covered by $\{V_n | n \in N\}$ and there exists a countable open cover $\{O_n \mid n \in \mathcal{N}\}$ of $X_{W'}$ such that $\overline{O_n}^{X_w} \subseteq V_n \cap X_{W'} = \cup\{V_{n\alpha} \cap X_{W'} | \alpha \in \Delta\}$ for every $n \in \mathbb{N}$ .For every $\alpha \in \Delta$,let $M_\alpha = \cup\{O_n \cap (\cup\{V_{n\beta} \cap X_{W'} | \beta \le \alpha\}) | n \in N\}$,then $\{M_\alpha \mid \alpha \in \Delta\}$is a monotone increasing open cover of $X_{W'}$ such that $\overline{M_\alpha}^{X_{W'}} \subseteq U_\alpha \cap X_{W'}$. Therefore X has $\mathfrak{B}$-Property fibrewisely.

Theorem 2.5: If a fibrewise topological space X over B has $\mathfrak{B}$-Property fibrewisely then for every $b \in B$ and every open (in X) cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ of X_b, there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and there exists an open (in X) cover $\mathcal{V} = \{V_{\alpha\beta} | \beta \le \alpha; \alpha \in \Delta\}$of X_w such that :

(1) $V_{\alpha\beta} \subseteq U_\beta \cap X_W$ for any α, β with $\beta \le \alpha$.

(2) For each $x \in X_w$, there exists a nbhd O of x and $\alpha_x \in \Delta$ such that $O \cap (\cup\{V_{\alpha\beta} | \beta \le \alpha; \alpha \ge \alpha_x\}) = \phi$.

Proof: Let $b \in B$ and $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be an open (in X) cover of X_b. Let $U'_\alpha = \cup\{U_\beta | \beta \le \alpha\}$. Then $\mathcal{U}' = \{U'_\alpha \mid \alpha \in \Delta\}$ is a monotone increasing open cover of X_b. Since X has $\mathfrak{B}$-Property fibrewisely, there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}'$ and there exists a monotone increasing open cover $\mathcal{V}=\{V_\alpha \mid \alpha \in \Delta\}$ of X_W such that $\overline{V_\alpha}^{X_W} \subseteq U'_\alpha \cap X_W$ for each $\alpha \in \Delta$. Let $V_\alpha = \bigcup\{V_\beta | \beta \leq \alpha\}$ for each $\alpha \in \Delta$ and for each $\alpha, \beta \in \Delta$ with $\beta \leq \alpha$, let $V_{\alpha\beta} = (U_\beta \cap X_W) \setminus \overline{V_{\alpha-1}}^{X_W}$. Then $V_{\alpha\beta} \subseteq U_\beta \cap X_W$ for each $\alpha, \beta \in \Delta$ with $\beta \leq \alpha$.

Let $x \in X_W$ and α_0 be the first of $\{\alpha \in \Delta | x \in U'_\alpha\}$, then $x \notin U'_{\alpha-1}$, so $x \notin \overline{V_{\alpha_0-1}}^{X_W}$. Since $U'_{\alpha_0} = \bigcup\{V_\beta | \beta \leq \alpha_0\}$, then there exists $\beta \leq \alpha_0$ such that $x \in U_\beta$, so $x \in (U_\beta \cap X_W))\setminus \overline{V_{\alpha_0-1}}^{X_W} = V_{\alpha_0\beta}$, and so $\{V_{\alpha\beta} | \beta \leq \alpha; \alpha \in \Delta\}$ covers of X_W.

Let $x \in X_W$ and $\alpha_0 \in \Delta$ with $x \in V_{\alpha_0}$. For any $\alpha \in \Delta$ with $\alpha_0 \leq \alpha$, we have $V_{\alpha_0} \cap (X_W \setminus \overline{V_\alpha}^{X_W}) = \phi$ because $\{V_\alpha \mid \alpha \in \Delta\}$ is monotone increasing, so for any $\alpha \in \Delta$ with $\alpha \geq \alpha_0 + 1$ and any $\beta \in \Delta$ with $\beta \leq \alpha$ we have $V_{\alpha_0} \cap V_{\alpha\beta} \subseteq V_{\alpha_0} \cap (X_W \setminus \overline{V_{\alpha-1}}^{X_W}) = \phi$ so $V_{\alpha_0} \cap (\bigcup\{V_{\alpha\beta} | \beta \geq \alpha; \alpha \geq \alpha_0 + 1\} = \phi$, hence $\{V_{\alpha\beta} | \beta \leq \alpha; \alpha \in \Delta\}$ satisfies (2).

Theorem 2.6: Let *X* be a fibrewise topological space over *B*. Then the following are equivalent:

(1) *X* has $\mathfrak{B}$-Property fibrewisely.

(2) For every $b \in B$ and every monotone increasing open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ of X_b, there exists a nbhd *W* of *b* in *B* such that X_W is covered by $\mathcal{U}$ and there exists an open (in *X*) cover $\{V_\alpha \mid \alpha \in \Delta\}$ of X_W such that:

   (a) $V_\alpha \subseteq U_\alpha \cap X_W$ for each $\alpha \in \Delta$.

   (b) For each $x \in X_W$ there exists a nbhd O_x of *x* in X_W and some $\alpha_x \in \Delta$ such that $O_x \cap (\bigcup\{V_\alpha | \alpha \geq \alpha_x\})$.

(3) For every $b \in B$ and every infinite open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ of of X_b there exists a nbhd *W* of *b* in *B* such that X_W is covered by $\mathcal{U}$ and $\{U_\alpha \cap X_W \mid \alpha \in \Delta\}$ has an open refinement $\mathcal{V}$ such that each $x \in X_W$ has a nbhd O_x such that the cardinality of $\{V \in \mathcal{V} | O_x \cap V \neq \phi\}$ is less than the cardinality of $\mathcal{U}$.

Proof: (1)$\rightarrow$(2) Let $b \in B$ and $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a monotone increasing open cover of X_b. Then there exists a nbhd *W* of *b* in *B* such that X_W is covered by $\mathcal{U}$ and there exists a monotone increasing open cover $\{T_\alpha \mid \alpha \in \Delta\}$ of X_W in X_W such that $\overline{T_\alpha}^{X_W} \subseteq U_\alpha \cap X_\alpha$ for each $\alpha \in \Delta$. Since $\{T_\alpha \mid \alpha \in \Delta\}$ is an open cover of X_W then there exists a nbhd $W' \subseteq W$ of *b* in *B* such that $X_{W'}$ is covered by $\{T_\alpha \mid \alpha \in \Delta\}$ and there exists monotone increasing open cover $\{S_\alpha \mid \alpha \in \Delta\}$ such that $\overline{S_\alpha}^{X_{W'}} \subseteq T_\alpha \subseteq \overline{T_\alpha}^{X_{W'}} \subseteq U_\alpha \cap X_W$ for each $\alpha \in \Delta$. Assume that $T_\alpha = \bigcup\{T_\beta | \beta \leq \alpha\}$ for any $\alpha \in \Delta$ and let $V_\alpha = T_\alpha \setminus \overline{S_{\alpha-1}}^{X_{W'}}$ for each $\alpha \in \Delta$.

Let $x \in X_{W'}$ and α_0 be the first of $\{\alpha \in \Delta \mid x \in T_\alpha\}$.Then $x \notin T_{\alpha_0 - 1}$ and hence $x \notin \overline{S_{\alpha_0-1}}^{X_{W'}}$, so $x \in V_{\alpha_0}$ Thus $\{V_\alpha \mid \alpha \in \Delta\}$ is an open cover of $X_{W'}$.

Let $x \in X_{W'}$. Since $\{S_\alpha \mid \alpha \in \Delta\}$ covers $X_{W'}$, there exists $\alpha_0 \in \Delta$ such that $x \in S_{\alpha_0}$, so for every $\alpha \geq \alpha_0$, we have $S_{\alpha_0} \cap V_\alpha \subseteq S_{\alpha_0} \setminus \overline{S_{\alpha-1}}^{X_{W'}} \subseteq S_{\alpha_0} \setminus \overline{S_{\alpha_0}}^{X_{W'}} = \phi$.Therefore $S_{\alpha_0} \cap (\bigcup\{V_\alpha | \alpha \geq \alpha_0\})$.

(2) $\to$(1) Let $b \in B$ and $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a monotone increasing open cover of X_b. Then there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and there exists an open (in X) cover $\mathcal{V}=\{V_\alpha \mid \alpha \in \Delta\}$ of X_w in X_w such that $\mathcal{V}$ satisfies (a) and (b) of (2).For each $\alpha \in \Delta$ let $T_\alpha = \bigcup \{O \mid O$ is open in X_w and $O \cap (\bigcup \{V_\beta | \beta \geq \alpha\}) = \phi\}$. Then $\{T_\alpha \mid \alpha \in \Delta\}$ is a monotone increasing open cover of X_w.For each $\alpha \in \Delta$, $T_\alpha \cap (\bigcup \{V_\beta | \beta \geq \alpha\}) = \phi\})$, and so $\overline{T_\alpha}^{X_w} \cap (\bigcup\{V_\beta \mid \beta \geq \alpha\}) = \phi$, hence $\overline{T_\alpha}^{X_w} \subseteq X_W \setminus \bigcup\{V_\beta \mid \beta \geq \alpha\} \subseteq \bigcup\{V_\beta \mid \beta < \alpha\} \subseteq \bigcup\{U_\beta \cap X_W \mid \beta < \alpha\} \subseteq U_\alpha \cap X_W$.

Therefore X has $\mathscr{B}$-Property fibrewisely.

(3) $\to$ (2) Clear.

(1)$\to$(3) Let $b \in B$ and $\mathcal{U} = \{U_\alpha \mid \alpha < \tau\}$ be an open (in X) cover of X_b, where τ is the minimal ordinal number whose cardinality is equal to the cardinality of $\mathcal{U}$. By theorem 2.5, there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and there exists an open (in X) cover $\mathcal{V} = \{V_{\alpha\beta} \mid \beta \leq \alpha; \alpha < \tau\}$ of X_w such that:

(1) $V_{\alpha\beta} \subseteq U_\beta \cap X_W$ for any α, β with $\beta \leq \alpha$ and

(2) For each $x \in X_w$ there exists a nbhd O of x and $\alpha_x < \tau$ such that $O \cap (\bigcup\{V_{\alpha\beta} \mid \beta \leq \alpha; \alpha_x < \alpha\}) = \phi$, so the cardinality of $\{V \in \mathcal{V} \mid O \cap V \neq \phi\}$ is less than the cardinality of $\mathcal{U}$.

From theorem 2.5 and theorem 2.6 we have

Theorem 2.7: Let X be a fibrewise topological space over B. Then the following are equivalent:

1. X has $\mathscr{B}$-Property fibrewisely.
2. For every $b \in B$ and every monotone increasing open (in X) cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ of X_b, there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and there exists an open (in X) cover $\mathcal{V} = \{V_{\alpha\beta} \mid \beta \leq \alpha; \alpha \in \Delta\}$ of X_w such that:

(a) $V_{\alpha\beta} \subseteq U_\beta \cap X_W$ for any α, β with $\beta \leq \alpha$.

(b) For each $x \in X_w$, there exists a nbhd O of x and $\alpha_x \in \Delta$ such that $O \cap (\bigcup\{V_{\alpha\beta} \mid \beta \leq \alpha; \alpha_x < \alpha\}) = \phi$.

Proof: (1)$\to$(2) From theorem 2.5.

(2)$\rightarrow$(1) Let $b\in B$ and $\mathcal{U}=\{U_\alpha \mid \alpha\in\Delta\}$ be a monotone increasing open cover of X_b. Then There exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and there exists an open (in X) cover $\mathcal{V}$ = $\{V_{\alpha\beta} \mid \beta\le\alpha;\alpha\in\Delta\}$ of X_w satisfying (a) and (b). Let $V_\alpha=\{V_{\alpha\beta}\mid\beta\le\alpha\}$ for each $\alpha\in\Delta$, then $\{V_\alpha \mid \alpha\in\Delta\}$ is an open cover of X_w such that $V_\alpha\subseteq U_\alpha$ for each $\alpha\in\Delta$. Also for each $x\in X_w$ there exists a nbhd O of x and $\alpha_x\in\Delta$ such that $O\cap(\bigcup\{V_{\alpha\beta}\mid\beta\le\alpha;\alpha_x\le\alpha\})=\phi$, so $O\cap(\bigcup\{V_\alpha\mid\alpha_x\le\alpha\})=\phi$. Therefore, from theorem 2.6 (2), X has $\mathcal{B}$-Property fibrewisely.

Theorem 2.8: A fibrewise regular (T_3) space X over B is fibrewise Lindelof if and only if X has $\mathcal{B}$-Property fibrewisely and for every $b\in B$,every uncountable subset of X_b has a limit point in X_b .

Proof: Suppose that X is fibrewise Lindelof over B. Then, by theorem 1.15, X is fibrewise paracompact over B, so X has $\mathcal{B}$-Property fibrewisely. Let $b\in B$ and A be an uncountable subset of X_b , and suppose that A has no limit point in X_b ,then for any $x\in X_b$ there exists a nbhd U_x of x such that $U_x\cap(A\setminus\{x\})=\phi$, so $\{U_x\mid x\in X_b\}$ is an open cover of X_b which has no countable subcover, a contradiction. Thus A has a limit point in X_b .

Conversely, suppose that X is fibrewise Lindelof over B, let $b\in B$ and let $\mathcal{U}=\{U_\alpha \mid \alpha\in\Delta\}$ be a monotone increasing open (in X) cover of X_b ,then there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and there exists a monotone increasing open (in X) cover $\{V_\alpha \mid \alpha\in\Delta\}$ of X_w such that $\overline{V_\alpha}^{X_w}\subseteq U_\alpha$ Suppose that $\mathcal{U}$ has no countable subcover of X_b ,then $X_b\setminus U_\alpha$ is an uncountable subset of X_b , for each $\alpha\in\Delta$, hence $X_b\cap(U_{\alpha+1}\setminus U_\alpha)\neq\phi$ for an uncountable number of α 's. Let $\Delta'=\{\alpha\in\Delta\mid(U_{\alpha+1}\setminus U_\alpha)\cap X_b\neq\phi\}$ and $x_\alpha\in(X_b\setminus U_\alpha)\cap(U_{\alpha+1}\setminus U_\alpha)$ for all $\alpha\in\Delta'$. Then $A=\{x_\alpha\mid\alpha\in\Delta'\}$ is an uncountable subset of X_b , so it has a limit point x. Let β be the first of $\{\alpha\in\Delta'\mid x\in U_\alpha\}$. Then $U_{\mathcal{B}}\setminus\overline{V_{\beta-1}}^{X_w}$ is a nbhd of x with $(U_\alpha\setminus\overline{V_{\beta-1}}^{X_w})\cap(A\setminus\{x\})$ is at most finite. Since X is fibrewise T_1 then $(U_\alpha\setminus\overline{V_{\beta-1}}^{X_w})\cap A$ is an infinite set, a contradiction, so $\mathcal{U}$ has a countable subcover of X_b. Therefore X is fibrewise Lindelof.

Corollary 2.9: A fibrewise countably compact and fibrewise T_3 space X over B is fibrewise compact if and only if X has $\mathcal{B}$-Property fibrewisely.

Proof: suppose that X is fibrewise countably compact and fibrewise T_3 over B. Then for any $b\in B$, every infinite countable subset of X_b has a limit point, so every uncountable subset of X_b has a limit point, so if X has $\mathcal{B}$-Property fibrewisely over B, then ,by theorem 2.8, X is fibrewise Lindelof over B. Thus X is fibrewise compact over B.

The converse is clear.

Definition 2.10: A fibrewise topological space X is called fibrewise Para-Lindelof if for every $b \in B$ and every open (in X) cover $\mathcal{U}=\{U_\alpha \mid \alpha \in \Delta\}$ of X_b, there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and $\{U_\alpha \cap X_w \mid \alpha \in \Delta\}$ has a locally countable open (in X) refinement that covers X_w.

Theorem 2.11: If a fibrewise topological space X is fibrewise Para-Lindelof and fibrewise countably paracompact over B, then X has $\mathscr{B}$-Property fibrewisely over B.

Proof: Let $b \in B$ and $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a monotone increasing open (in X) cover of X_b. Then we have the following cases:

Case1: If $\mathcal{U}$ is uncountable, then there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and $\{U_\alpha \cap X_w \mid \alpha \in \Delta\}$ has a locally countable open (in X) refinement $\mathcal{V}= \{V_\alpha \mid \alpha \in \Delta\}$ that covers X_w with $V_\alpha \subseteq U_\alpha \cap X_w$. For each $x \in X_w$ there exists a nbhd O_x of x in X_w that intersects only countably many members of $\mathcal{V}$, so there exists $\alpha_x \in \Delta$ such that $O_x \cap V_\alpha = \phi$ for all $\alpha > \alpha_x$, hence $O_x \cap (\bigcup\{V_\alpha \mid \alpha_x < \alpha\}) = \phi$. Therefore, by theorem 2.6, X has $\mathscr{B}$-Property fibrewisely over B.

Case2: If $\mathcal{U}$ is countable then, by theorem 1.17, there exists a nbhd W of b in B such that X_w is covered by $\mathcal{U}$ and $\{U_\alpha \cap X_w \mid \alpha \in \Delta\}$ has a locally finite open (in X) refinement $\mathcal{V}=\{V_\alpha \mid \alpha \in \Delta\}$ that covers X_w with $V_\alpha \subseteq U_\alpha \cap X_W$. For each $x \in X_w$ there exists a nbhd O_x of x in X_w that intersects only finitely many members of $\mathcal{V}$, hence there exists $\alpha_x \in \Delta$ such that $O_x \cap V_\alpha = \phi$ for all $\alpha > \alpha_x$, so $O_x \cap (\bigcup\{V_\alpha \mid \alpha_x < \alpha\}) = \phi$. Therefore, by theorem 2.6, X has $\mathscr{B}$-Property fibrewisely over B.

Properties of Fibrewise $\mathscr{B}$-Property

Theorem 3.1: Let X be a fibrewise topological space over B. Then if X has $\mathscr{B}$-Property fibrewisely over B and Y is a closed subspace of X, then Y has $\mathscr{B}$-Property fibrewisely over B.

Theorem 3.2: Let X be a fibrewise T_2 space over B. If X has $\mathscr{B}$-Property fibrewisely over B, then X is fibrewise T_3 over B.

Proof: Let $b \in B$ and suppose that there exists a closed set H in X and a point $x \in X_b$ with $x \notin H$ such that if W is any nbhd of b in B and O is any open set in X_w with $H \cap X_W \subseteq O$ then $x \in \overline{O}^{X_W}$ .Since X is fibrewise T_2 then for every $y \in H_b$ there exists a nbhd U_y of y such that $x \notin \overline{U_y}$, so $x \notin \overline{U_y} \cap X_W = \overline{U_y}^{X_W}$ for any nbhd W of b in B, hence we can find an open cover $\mathcal{U}$ of H_b of minimal cardinal λ such that if $U \in \mathcal{U}$ then $x \notin \overline{U}^{X_W}$

for any nbhd *W* of *b* in *B*. Let $\{U_\alpha \mid \alpha\in\Delta\}$ be a well-ordering of $\mathcal{U}$ according to the initial ordinal of cardinal λ.Note that λ cannot be finite because if so, then $\mathcal{U}=\{U_1,...,U_n\}$, $x\notin \overline{U_i}^{X_W}$, *i=1,2,….,n* for any nbhd *W* of b in *B*. Since *H* has the $\mathcal{B}$-Property fibrewisely and we can make $\mathcal{U}$ a monotone increasing open cover of H_b, then there exists a nbhd *M* of *b* in *B* such that $H_M = H\cap X_M$ is covered by $\mathcal{U}$, so $H\cap X_M \subseteq \bigcup_{i=1}^{n} U_i$ .Since $\overline{\bigcup_{i=1}^{n} U_i}^{X_M} = \bigcup_{i=1}^{n}\overline{U_i}^{X_M}$ then $x\notin \overline{\bigcup_{i=1}^{n} U_i}^{X_M}$,so $x\notin \overline{\bigcup_{i=1}^{n} U_i\cap X_M}^{X_M}$ but $x\in \overline{O}^{X_M}$ for any nbhd *M* of *b* in *B* and any open set *O* in X_M with $H\cap X_M\subseteq O$,a contradiction, so λ is infinite.

Let $V_\alpha = \bigcup_{\beta\le\alpha} U_\beta$, then $\{V_\alpha \mid \alpha\in\Delta\}$ is a monotone increasing open cover of H_b , so there exists a nbhd W' of b such that $H_{W'}$ is covered by $\{V_\alpha \mid \alpha\in\Delta\}$ and there exists a monotone increasing open cover $\{G_\alpha \mid \alpha\in\Delta\}$ of $H_{W'}$ such that $\overline{G_\alpha}^{H_{W'}} \subseteq V_\alpha\cap H_{W'}$ .Let $\alpha_0\in\Delta$ such that $x\notin \overline{V_{\alpha_0}}^{H_{W'}}$, then $\alpha_0<\lambda$ and $x\notin \overline{G_{\alpha_0}}^{H_{W'}}$ .Let $G=\bigcup_{\alpha\ge\alpha_0} G_\alpha$,then $\mathcal{V}=\{U_\beta \mid \beta\le\alpha_0\}\cup\{G\}$ is an open cover of $H_{W'}$ such that the cardinality of $\mathcal{V}$ is strictly less than λ, which is a contradiction.

Therefore *X* is fibrewise T_3 over *B*.

Recall that a function $f: X\to Y$ is called proper if *f* is continuous, closed and $f^{-1}(y)$ is compact for any $y\in Y$.

Theorem 3.3: Let *X* and *Y* be fibrewise a topological spaces over *B* and $f: X\to Y$ be a proper fibrewise surjection. Then *X* has $\mathcal{B}$-Property fibrewisely over *B* if and only if *Y* has $\mathcal{B}$-Property fibrewisely over *B*.

Proof: suppose that *X* has $\mathcal{B}$-Property fibrewisely over *B* and let $\mathcal{U}=\{U_\alpha \mid \alpha\in\Delta\}$ be a monotone increasing open cover of Y_b ,then $\{f^{-1}(U_\alpha) \mid \alpha\in\Delta\}$ is a monotone increasing open cover of X_b ,so there exists a nbhd *W* of b in *B* such that X_w is covered by $\{f^{-1}(U_\alpha) \mid \alpha\in\Delta\}$ and there exists an open cover $\{V_\alpha \mid \alpha\in\Delta\}$ of X_w such that $V_\alpha\subseteq f^{-1}(U_\alpha)\cap X_W$ and for each $x\in X_w$ there exists a nbhd O_x of *x* and $\alpha_x\in\Delta$ such that $O_x\cap(\bigcup\{V_\alpha \mid \alpha\ge\alpha_x\})=\phi$.

Let $O_\alpha = Y\setminus f(X\setminus V_\alpha)$, then O_α is open in *Y* and $O_\alpha\subseteq U_\alpha\cap Y_W$ and if $y\in Y_w$ then there exists $x\in f^{-1}(y)$, so there exists a nbhd O_x of *x* and $\alpha_x\in\Delta$ such that $O_x\cap(\bigcup\{V_\alpha \mid \alpha\ge\alpha_x\})=\phi$.

Let $M_y = Y\setminus f(X\setminus O_x)$, then $M_y\cap(\bigcup\{O_\alpha \mid \alpha\ge\alpha_x\})=\phi$.Thus, by theorem 2.6, *Y* has $\mathcal{B}$-Property fibrewisely.

Conversely, suppose that *Y* has $\mathcal{B}$-Property fibrewisely over *B* and let $\mathcal{U}=\{U_\alpha \mid \alpha\in\Delta\}$ be a monotone increasing open cover of X_b. Since *f* is proper then $f^{-1}(y)$ is compact, so there exists $\alpha(y) \in \Delta$ such that $f^{-1}(y) \subseteq U_{\alpha(y)}$.Also since f is closed then there exists a nbhd V_y of y in Y such that $f^{-1}(V_y) \subseteq U_{\alpha(y)}$.For each $\alpha \in \Delta$ let $G_\alpha = \bigcup\{V_y \mid f^{-1}(V_y) \subseteq U_{\alpha(y)}, y \in Y_b\}$.Then $\mathcal{G} = \{G_\alpha \mid \alpha \in \Delta\}$is a monotone increasing open cover of Y_b, so there exists a nbhd W of b in B such that Y_w is covered by $\mathcal{G}$ and there exists a monotone increasing open cover $\{O_\alpha \mid \alpha \in \Delta\}$of Y_w such that $\overline{O_\alpha}^{Y_w} \subseteq G_\alpha \cap Y_W$, so $\overline{f^{-1}(O_\alpha)}^{X_W} \subseteq f^{-1}(\overline{O_\alpha}^{Y_W}) \subseteq f^{-1}(G_\alpha) \cap X_W \subseteq U_\alpha \cap X_W$.

Therefore X has 𝔅-Property fibrewisely over B.

Corollary 3.4: If X is fibrewise compact over B and Y has ***𝓑***-Property fibrewisely over B. Then $X \times_B Y$ has 𝔅-Property fibrewisely over B.

Proof: The fibrewise projection $\pi_Y : X \times_B Y \to Y$ is a proper fibrewise surjection over B.

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