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A Study of Certain Bundles of a Differentiable Manifold

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Abstract

The bundles of a differentiable manifold are also differentiable manifolds of higher dimensions. In the present paper, we study certain bundles like cotangent bundle, tangent bundle, linear frame bundle, principal fibre bundle etc. It has also been shown that a pentalinear frame bundle is the principal fibre bundle.

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Introduction

Let M be a C^∞ manifold of dimension n. At each point $P \in M$, there is associated an n-dimensional vector space of tangent vectors called tangent space. We represent the tangent space at P of M by T_p(M) and $U_{P\in M}\ T_p(M) = T(M)$, $T(M)$ is called tangent bundle of M. If M is a differentiable manifold of dimension n, $T(M)$ is a differentiable manifold of dimension $2n$. Let π be projection map $T(M) \rightarrow M$. Let $P \in M$ in the coordinate neighborhood U with local coordinate system (x^h). Then $\pi^{-1}(U)$ is a coordinate neighborhood in $T(M)$ with local coordinates (x^h, y^h) where $y^h = \dot{x}^h$ are components of a contravarient vector.

If $U(x^h)$ and $U'(h')$ be two coordinate neighborhoods in M such that $U \cap U' \neq \emptyset$, then

$\pi^{-1}(U)$ and $\pi^{-1}(U')$ are also not disjoint in $T(M)$.

(i) $x^{h'} = x^{h'}(x^h)$ (ii) $y^{h'} = \frac{\partial x^{h'}}{\partial x^h} y^h$ (1.1)

At each point $P \in M$, there is associated an n-dimensional vector space $T_p{}^*(M)$ dual to tangent space $T_p(M)$.

Then $\cup_{p\in M} T_p{}^*(M) = T^*(M)$, $T^*(M)$ is called cotangent bundle of M. Let π be the projection map $T^*(M) \to M$.

At each point $P \in M$ in coordinate neighborhood $U, \pi^{-1}(U)$ has local coordinates (x^h, p_i) where p_i are components of $1-$form. If $\pi^{-1}(U) \cap \pi^{-1}(U) \neq \emptyset$, then at a point in the intersection region:

(i) $x^{h'} = x^{h'}(x^h)$ (ii) $p_{i'} = \frac{\partial x^i}{\partial x^{i'}} p_i$ (1.2)

Lifts of $(1,1)$ tensor field M in $T^*(M)$

Let us call the manifold M as base space. Consider base space admits a tensor field f of type (1, 1) satisfying the equation:

$$f^8 + \lambda^2 f^6 + \mu^2 f^4 = 0 \tag{2.1}$$

The complete lift f^c of f is a (1, 1) tenser field in $T^*(M)$ with local components [6]

$$f^c = \begin{bmatrix} f_i{}^h & 0 \\ p_a\left(\frac{\partial f_h^a}{\partial x^i} - \frac{\partial f_i^a}{\partial x^h}\right) & f_h{}^i \end{bmatrix} \tag{2.2}$$

If we put $p_a\left(\frac{\partial f_h^a}{\partial x^i} - \frac{\partial f_i^a}{\partial x^h}\right) = p_a \partial\, [\mathrm{i} f_h^a]$.

Then

$$(f^c) = \begin{bmatrix} f_i{}^h & 0 \\ p_a \partial\, [\mathrm{i} f_h^a] & f_h^i \end{bmatrix} \tag{2.3}$$

Hence

$$(f^c)^2 = \begin{bmatrix} f_i{}^h & 0 \\ p_a \partial\, [\mathrm{i}\, f_h^a] & f_h^i \end{bmatrix} \begin{bmatrix} f_j{}^i & 0 \\ p_t \partial\, [\mathrm{j} t_i^t] & f_i{}^j \end{bmatrix}$$

$$(f^c)^2 = \begin{bmatrix} f_i{}^h f_j{}^i & 0 \\ p_a f_j{}^i \partial[\mathrm{i} f_h^a] + p_t f_h{}^i \partial[\mathrm{j} t_i^t] & f_i{}^j f_h{}^i \end{bmatrix}$$

If we put $p_a f_j{}^i \partial[\mathrm{i} f_h^a] + p_t f_h{}^i \partial[\mathrm{j} t_i^t] - L_{hj}$,

Thus $(f^c)^2 = \begin{bmatrix} f_i{}^h f_j{}^i & 0 \\ L_{hj} & f_i{}^j f_h{}^i \end{bmatrix}$

$$(f^c)^4 = \begin{bmatrix} f_i^{\,h} f_j^{\,i} & 0 \\ L_{hj} & f_i^{\,j} f_h^{\,i} \end{bmatrix} \begin{bmatrix} f_k^{\,j} f_l^{\,k} & 0 \\ L_{jl} & f_k^{\,l} f_j^{\,k} \end{bmatrix}$$

If we put $f_k^{\,j} f_l^{\,k} L_{hj} + f_i^{\,j} f_h^{\,i} L_{jl} = R_{hl}$, then

$$(f^c)^4 = \begin{bmatrix} f_i^{\,h} f_j^{\,i} f_k^{\,j} f_l^{\,k} & 0 \\ R_{hl} & f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} \end{bmatrix}$$

$$(f^c)^6 = \begin{bmatrix} f_i^{\,h} f_j^{\,i} f_k^{\,j} f_l^{\,k} & 0 \\ R_{hl} & f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} \end{bmatrix} \begin{bmatrix} f_m^{\,l} f_n^{\,m} & 0 \\ L_{ln} & f_m^{\,n} f_l^{\,m} \end{bmatrix}$$

Now

$$(f^c)^6 = \begin{bmatrix} f_i^{\,h} f_j^{\,i} f_k^{\,j} f_l^{\,k} f_m^{\,l} f_n^{\,m} & 0 \\ f_m^{\,l} f_n^{\,m} R_{hl} + f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} L_{hn} & f_m^{\,n} f_l^{\,m} f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} \end{bmatrix}$$

Put $f_m^{\,n} f_n^{\,m} R_{hl} + f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} L_{hn} = S_{hn}$.

Thus $(f^c)^6 = \begin{bmatrix} f_i^{\,h} f_j^{\,i} f_k^{\,j} f_l^{\,k} f_m^{\,l} f_n^{\,m} & 0 \\ S_{hn} & f_m^{\,n} f_l^{\,m} f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} \end{bmatrix}$

$$(f^c)^8 = \begin{bmatrix} f_i^{\,h} f_j^{\,i} f_k^{\,j} f_l^{\,k} & 0 \\ R_{hl} & f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} \end{bmatrix} \begin{bmatrix} f_m^{\,l} f_n^{\,m} f_o^{\,n} f_p^{\,o} & 0 \\ R_{lp} & f_o^{\,p} f_n^{\,o} f_m^{\,n} f_l^{\,m} \end{bmatrix}$$

$$(f^c)^8 = \begin{bmatrix} f_i^{\,h} f_j^{\,i} f_k^{\,j} f_l^{\,k} f_m^{\,l} f_n^{\,m} f_o^{\,n} f_p^{\,o} & 0 \\ f_m^{\,l} f_n^{\,m} f_o^{\,n} f_p^{\,o} R_{hl} + f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} R_{lp} & f_o^{\,p} f_n^{\,o} f_m^{\,n} f_l^{\,m} f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} \end{bmatrix}$$

Put $f_m^{\,l} f_n^{\,m} f_o^{\,n} f_p^{\,o} R_{hl} + f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} R_{lp} = T_{hp}$

$$(f^c)^8 = \begin{bmatrix} f_i^{\,h} f_j^{\,i} f_k^{\,j} f_l^{\,k} f_m^{\,l} f_n^{\,m} f_o^{\,n} f_p^{\,o} & 0 \\ T_{hp} & f_o^{\,p} f_n^{\,o} f_m^{\,n} f_l^{\,m} f_k^{\,l} f_j^{\,k} f_i^{\,j} f_h^{\,i} \end{bmatrix}$$

So $(f^c)^8 + \lambda^2 (f^c)^6 + \mu^2 (f^c)^4 = 0$ iff

$T_{hp} + \lambda^2 S_{hn} + \mu^2 R_{hl} = 0$

Thus we have

Theorem (2.1) In order that the complete lift f^c of a (1, 1) tensor field f in M satisfying $f^8 + \lambda^2 f^6 + \mu^2 f^4 = 0$ may have similar structure in $T^*(M)$, it is necessary and sufficient that

$T_{hp} + \lambda^2 S_{hn} + \mu^2 R_{hl} = 0$

Pentalinear Frame Bundle

In this section, first of all, we have some definitions.

Definition (3.1) A set G is said to be a Lie group if G is a group as well as a differentiable manifold and two maps

$G \times G \longrightarrow G$ such that $(g_1, g_2) \longrightarrow g_1\, g_2$, $g_1, g_2 \in G$

and

$G \longrightarrow G$ such that $g \longrightarrow g^{-1}$, $g \in G$ are differentiable.

If $Gl(n,R)$ be the set of all $n \times n$ non-singular matrices over R, then $Gl(n,R)$ is a group under matrix multiplication. Moreover if $g \in Gl(n,R)$, we can write $g = (g_b^a)$, $g_b^a \in R$, $a,b = 1,2 \ldots\ldots n$. These n^2 real numbers g_b^a can be treated as coordinates and induce the manifold structure and $Gl(n,R)$ is a Lie Group. We say $Gl(n,R)$ as the general linear group.

Definition: (3.2) A set (P, M, G, π) is called principal fibre bundle if P is a differentiable manifold, G a Lie group and

G acts on P differentiably to the right. Hence there exists a differentiable map

$P \times G \longrightarrow P$ such that $(u, g) \longrightarrow ug$ $u \in P$, $g \in G$ and $ug \in P$ and

$(ug)h = u(gh)$, $g, h \in G$

M is the quotient manifold P/G and the map π: $P \longrightarrow M$ is differentiable.

For each $x \in M$ and for every coordinate neighborhood U of x, $\pi^{-1}(U)$ is isomorphic to $U \times G$.

Let M_1, M_2, M_3, M_4 & M_5 be five C^∞ manifold each of dimension n. If $x \in M_1$, $y \in M_2$, $z \in M_3$, $t \in M_4$, $s \in M_5$, it implies that (x,y,z,t,s) belongs to the cartesian product manifold $M_1 \times M_2 \times M_3 \times M_4 \times M_5$. Let $(x^1, x^2 \ldots x^n)$ be local coordinate system about x in M_1, $(y^1, y^2 \ldots y^n)$ about y in M_2, $(z^1, z^2 \ldots z^n)$ about z in M_3, $(t^1, t^2 \ldots t^n)$ about t in M_4, $(s^1, s^2 \ldots s^n)$ and about s in M_5. Then $\{x^i, y^j, z^k, t^l, s^m\}$ is local coordinate system about (x,y,z,t,s) in the product manifold. The canonical basis vectors about x in M_1 are $\frac{\partial}{\partial x^i}$, about y in M_2 are $\frac{\partial}{\partial y^j}$, about z in M_3 are $\frac{\partial}{\partial z^k}$, about t in M_4 are $\frac{\partial}{\partial t^l}$, and about s in M_5 are $\frac{\partial}{\partial s^m}$. Thus canonical basis vectors about (x,y,z,t,s) in the product manifold are $\left[\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial t^l}, \frac{\partial}{\partial s^m}\right) \leq i,j,k,l,m \leq n\right]$. We can write $X_a = X_a^i \frac{\partial}{\partial x^i}$, $Y_b = Y_b^j \frac{\partial}{\partial y^j}$, $Z_c = Z_c^k \frac{\partial}{\partial z^k}$, $T_d = T_d^l \frac{\partial}{\partial t^l}$ and $S_e = S_e^m \frac{\partial}{\partial s^m}$ thus the tangent vector at (x,y,z,t,s) of the product manifold $M_1 \times M_2 \times M_3 \times M_4 \times M_5$ is $(X_a, Y_b, Z_c, T_d, S_e)$. Let us say the set $(x^i, y^j, z^k, t^l, s^m, X_a^i, Y_b^j, Z_c^k, T_d^l, S_e^m)$ a pentalinear frame at (x,y,z,t,s) of the product manifold. If PL be set of all such pentalinear frames at different points of the product manifold, then it is easy to prove that PL is also a differentiable manifold.Let us call the set $\{PL, M_1 \times M_2 \times M_3 \times M_4 \times M_5, \pi, Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R)\}$

the Pentalinear Frame Bundle of the product manifold $M_1 \times M_2 \times M_3 \times M_4 \times M_5$.

Now we prove the following theorem:

Theorem (3.1) The Pentalinear Frame Bundle is the Principal Fibre Bundle.

Proof: To prove this, we have to show

$Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R)$ acts on PL differentiably to the right. We can take $u \in PL$ in the form $u = (x^i, y^j, z^k, t^l, s^m, U^i_a, V^j_b, W^k_c, T^l_d, S^m_e)$. If $g \in Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R)$,

$g = (A^a_l, B^b_m, C^c_n, D^d_q, E^e_r)$.

We can define a map

$$PL \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \longrightarrow PL$$

such that $\{(x^i, y^j, z^k, t^l, s^m, U^i_a, V^j_b, W^k_c, T^l_d, S^m_e)(A^a_l, B^b_m, C^c_n, D^d_q, E^e_r)\}$

$$\longrightarrow (x^i, y^j, z^k, t^l, s^m, U^i_a A^a_l, V^j_b B^b_m, W^k_c C^c_n, T^l_d D^d_q, S^m_e E^e_r) \in PL$$

and $(x^i, y^j, z^k, t^l, s^m, U^i_a A^a_l, V^j_b B^b_m, W^k_c C^c_n, T^l_d D^d_q, S^m_e E^e_r)(L^l_r, M^m_s, N^n_t, P^q_v, R^r_w)$

$$= (x^i, y^j, z^k, t^l, s^m, U^i_a, V^j_b, W^k_c, T^l_d, S^m_e)\,(A^a_l L^l_r, B^b_m M^m_s, C_n{}^c N^n_t, D^d_q P^q_v, E^e_r R^r_w)$$

Thus $Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R)$ acts on PL differentiably to the right.

(b) $M_1 \times M_2 \times M_3 \times M_4 \times M_5$ is the quotient manifold

$PL/Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R)$ and the projection map $\pi : PL \longrightarrow M_1 \times M_2 \times M_3 \times M_4 \times M_5$ is differentiable.

(c) For each $x \in M_1 \times M_2 \times M_3 \times M_4 \times M_5$ and for every neighborhood U of x with local co-ordinate system $(x^i, y^j, z^k, t^l, s^m)$ we can write

$$\pi^{-1}(U) = \{(x^i, y^j, z^k, t^l, s^m, U^i_a, V^j_b, W^k_c, T^l_d, S^m_e)\}.$$

Now,

$$U = \{(x^i, y^j, z^k, t^l, s^m)\}$$

$$Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) = \{(U^i_a, V^j_b, W^k_c, T^l_d, S^m_e)\}$$

So, $U \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R)$

$$= \{(x^i, y^j, z^k, t^l, s^m)\,(U^i_a, V^j_b, W^k_c, T^l_d, S^m_e)\}.$$

If I be the map

$I : \pi^{-1}(U) \longrightarrow U \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R)$

such that $(x^i, y^j, z^k, t^l, s^m, U_a^i, V_b^j, W_c^k, T_d^l, S_e^m) \longrightarrow \{(x^i, y^j, z^k, t^l, s^m), (U_a^i, V_b^j, W_c^k, T_d^l, S_e^m)\}$.

Then I is the identity map which is always an isomorphism. Thus $\pi^{-1}(U)$ is isomorphic to $U \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R) \times Gl(n,R)$.

So the pentalinear frame bundle is the principal fibre bundle.

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