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One Formula for Solution of the Linear Differential Equations of the Second Order with the Variable Coefficients

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Abstract

In this paper we obtained the formula for the common solution of the linear differential equation of the second order with the variable coefficients in the more common case. We also obtained the formula for the solution of the Cauchy problem.

Keywords: The linear differential equation, the second order, the variable coefficients, the formula for the common solution, Cauchy problem.

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We consider the equation

$$L[y] = y''+p(t)y'+q(t)y = f(t),\ t \in I, \qquad (1)$$

where $I = (t_1, t_2), t_1 < t_2$, $p(t), q(t)$ and *f(t)* are known continuous functions on *I*.

Many works [1-4] are dedicated to the determination of the common solutions of the linear and nonlinear ordinary differential equations. But in common case any formulas for the decision of the linear differential equations haven't obtained. It is well known that if *p(t)* = p_0 = *const*, *q(t)*= q_0 = *const*, then depending on the sign of discriminant $D = p_0^2 - 4q_0$ the common solution of the equation (1) will be written by three formulas. In this theme the equation (1) is investigated in the more common cases. Depending on the correlation between *p(t)* and *q(t)* formulas for the determination of the common solution of this equation were obtained.

Theorem 1. Let

$$q(t) = K^2(t) + \beta^2(t) + K'(t) - \frac{\beta'(t)}{\beta(t)}K(t), \tag{2}$$

$$K(t) = \frac{1}{2}\left[p(t) + \frac{\beta'(t)}{\beta(t)}\right], t \in I, \tag{3}$$

where $K'(t)$ and $\beta'(t)$ are respectively the derivatives of the functions $K(t)$ and $\beta(t)$, $\beta(t) \neq 0$ for all $t \in I$. Then the common solution of the equation (1) will be written in the next form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_3(t) \tag{4}$$

where c_1 and c_2 are arbitrary constants,

$$y_1(t) = exp\left\{-\int_{t_0}^{t} K(s)ds\right\} cos\left[\int_{t_0}^{t} \beta(s)ds\right], t_0 \in I \tag{5}$$

$$y_2(t) = exp\left\{-\int_{t_0}^{t} K(s)ds\right\} sin\left[\int_{t_0}^{t} \beta(s)ds\right]. \tag{6}$$

$$y_3(t) = \int_{t_0}^{t} exp\left\{-\int_{s}^{t} K(\tau)d\tau\right\}\frac{f(s)}{\beta(s)} sin\left[\int_{s}^{t} \beta(\tau)d\tau\right] ds. \tag{7}$$

Proof: We show that

$$L[y_1] = 0, L[y_2] = 0, L[y_3] = f(t), t \in I.$$

At first we proof $L[y_1] = 0$. In fact if we differentiate (5) we shall obtain

$$y_1'(t) = -K(t)y_1(t) - \beta(t)exp\left\{-\int_{t_0}^{t} K(s)ds\right\} sin\left[\int_{t_0}^{t} \beta(s)ds\right], \tag{8}$$

$$y_1''(t) = -K'(t)y_1(t) - K(t)y_1'(t) - \beta^2(t)y_1(t) -$$
$$-[\beta'(t) - \beta(t)K(t)]exp\left\{-\int_{t_0}^{t} K(s)ds\right\} sin\left[\int_{t_0}^{t} \beta(s)ds\right]. \tag{9}$$

Then taking into account (8),(9),(2)and (3) we have

$$-K'(t)y_1(t) - K(t)y_1'(t) - \beta^2(t)y_1(t) -$$

$$-[\beta'(t) - \beta(t)K(t)]exp\left\{-\int_{t_0}^{t} K(s)ds\right\} sin\left[\int_{t_0}^{t} \beta(s)ds\right] +$$

$$\left[2K(t) - \frac{\beta'(t)}{\beta(t)}\right] y_1'(t) + K^2(t)y_1(t) + \beta^2(t)y_1(t) + K'(t)y_1(t) - \frac{\beta'(t)}{\beta(t)} K(t)y_1(t) =$$

$$= \left[K(t) - \frac{\beta'(t)}{\beta(t)}\right] [y_1'(t) + K(t)y_1(t)] - [\beta'(t) - \beta(t)K(t)]exp\left\{-\int_{t_0}^{t} K(s)ds\right\} sin\left[\int_{t_0}^{t} \beta(s)ds\right] =$$

$$= \left[K(t) - \frac{\beta'(t)}{\beta(t)}\right] \left\{-K(t)y_1(t) - \beta(t)exp\left[-\int_{t_0}^{t} K(s)ds\right] sin\left[\int_{t_0}^{t} \beta(s)ds\right] + K(t)y_1(t)\right\} -$$

$$-[\beta'(t) - \beta(t)K(t)]exp\left[-\int_{t_0}^{t} K(s)ds\right] sin\left[\int_{t_0}^{t} \beta(s)ds\right] = 0, t \in I.$$

Thus it is proved $L[y_1] = 0$.

We show $L[y_1] = 0$. If we differentiate (6) we shall have

$$y_2'(t) = -K(t)y_2(t) + \beta(t)exp\left[-\int_{t_0}^{t} K(s)ds\right] cos\left[\int_{t_0}^{t} \beta(s)ds\right], \qquad (10)$$

$$y_2''(t) = -K'(t)y_2(t) - K(t)y_2'(t) - \beta^2(t)y_2(t) +$$

$$+[\beta'(t) - \beta(t)K(t)]exp\left[-\int_{t_0}^{t} K(s)ds\right] cos\left[\int_{t_0}^{t} \beta(s)ds\right]. \qquad (11)$$

On the strength of (10),(11), (2) and (3) it follows that

$$L[y_2] = -K'(t)y_2(t) - K(t)y_2'(t) - \beta^2(t)y_2(t) +$$

$$+[\beta'(t) - \beta(t)K(t)]exp\left[-\int_{t_0}^{t} K(s)ds\right] cos\left[\int_{t_0}^{t} \beta(s)ds\right] +$$

$$+\left[2K(t) - \frac{\beta'(t)}{\beta(t)}\right] y_2'(t) + K^2(t)y_2(t) + \beta^2(t)y_2(t) + K'(t)y_2(t) - K(t)\frac{\beta'(t)}{\beta(t)} y_2(t) =$$

$$= \left[K(t) - \frac{\beta'(t)}{\beta(t)}\right][y_2'(t) + K(t)y_2(t)]$$
$$+ [\beta'(t) - \beta(t)K(t)]exp\left[-\int_{t_0}^{t} K(s)ds\right] cos\left[\int_{t_0}^{t} \beta(\tau)d\tau\right] =$$
$$= \left[K(t) - \frac{\beta'(t)}{\beta(t)}\right]\left\{K(t)y_2(t) + \beta(t)exp\left[-\int_{t_0}^{t} K(\tau)d\tau\right] cos\left[\int_{t_0}^{t} \beta(\tau)d\tau\right]\right.$$
$$\left. + K(t)y_2(t)\right\} +$$
$$+[\beta'(t) - \beta(t)K(t)]exp\left[-\int_{t_0}^{t} K(s)ds\right] cos\left[\int_{t_0}^{t} \beta(\tau)d\tau\right] = 0, t \in I.$$

We are going to proof $L[y_3] = f(t), t \in I$. Differentiating (7) we have

$$y_3'(t) = -K(t)y_3(t) + \beta(t)\int_{t_0}^{t} exp\left[-\int_{s}^{t} K(\tau)d\tau\right]\frac{f(s)}{\beta(s)} cos\left[\int_{s}^{t} \beta(\tau)d\tau\right] ds, \quad (12)$$

$$y_3''(t) = -K'(t)y_3(t) - K(t)y_3'(t) - \beta^2(t)y_3(t) + f(t) +$$
$$+[\beta'(t) - K(t)\beta(t)]\int_{t_0}^{t} exp\left[-\int_{s}^{t} K(\tau)d\tau\right]\frac{f(s)}{\beta(s)} cos\left[\int_{s}^{t} \beta(\tau)d\tau\right] ds. \quad (13)$$

Taking into account (12), (13), (2) and (3) we have

$$L[y_3] = -K'(t)y_3(t) - K(t)y_3'(t) - \beta^2(t)y_3(t) + f(t) +$$
$$+[\beta'(t) - K(t)\beta(t)]\int_{t_0}^{t} exp\left[-\int_{s}^{t} K(\tau)d\tau\right]\frac{f(s)}{\beta(s)} cos\left[\int_{s}^{t} \beta(\tau)d\tau\right] ds +$$
$$+\left[2K(t) - \frac{\beta'(t)}{\beta(t)}\right] y_3'(t) + K^2(t)y_3(t) + \beta^2(t)y_3(t) + K'(t)y_3(t)$$
$$- K(t)\frac{\beta'(t)}{\beta(t)} y_3(t) =$$
$$= \left[K(t) - \frac{\beta'(t)}{\beta(t)}\right][y_3'(t) + K(t)y_3(t)] + f(t) + [\beta'(t) - K(t)\beta(t)] \times$$
$$\times \int_{t_0}^{t} exp\left[-\int_{s}^{t} K(\tau)d\tau\right]\frac{f(s)}{\beta(s)} cos\left[\int_{s}^{t} \beta(\tau)d\tau\right] ds = \left[K(t) - \frac{\beta'(t)}{\beta(t)}\right] \times$$
$$\times\left\{-K(t)y_3(t) + \beta(t)\int_{t_0}^{t} exp\left[-\int_{s}^{t} K(\tau)d\tau\right]\frac{f(s)}{\beta(s)} cos\left[\int_{s}^{t} \beta(\tau)d\tau\right] ds + K(t)y_3(t)\right\}$$
$$+$$

$$+f(t) + [\beta'(t) - K(t)\beta(t)] \int_{t_0}^{t} exp\left[-\int_{s}^{t} K(\tau)d\tau\right] \frac{f(s)}{\beta(s)} cos\left[\int_{s}^{t} \beta(\tau)d\tau\right] ds$$
$$= f(t), t \in I.$$

Then $L[c_1y_1(t) + c_2y_2(t) + y_3(t)] = c_1L[y_1] + c_2L[y_2] + L[y_3] = f(t),\ \ t \in I,$ where c_1, c_2 are arbitrary constants. Theorem 1 has been proved.

Corallary: Let $p(t), K(t) \in C(I)$,

$$q(t) = K(t)p(t) - K^2(t) + K'(t) + a^2 exp\left\{2\int [2K(t) - p(t)]dt\right\}, \tag{14}$$

where $a \in R, a \neq 0$. Then the common solution of the equation (1) will be written in the form (4), where the functions $y_1(t), y_2(t)$ and $y_3(t)$ are defined by the formulas (5), (6), (7) and

$$\beta(t) = a exp\left\{2\int [2K(t) - p(t)]dt\right\}, t \in I. \tag{15}$$

Proof: Differtiating (15) we obtain

$$\beta'(t) = [2K(t) - p(t)]\beta(t), t \in I.$$

Hence we have (3). Taking into account (15) and (3) we obtain (2). The corallary has been proved.

Theorem 2: Let $t_0 \in I = (t_1, t_2)$, and suppose that the conditions of Theorem 1 hold. Then solution of the equation (1) with initial condition

$$y(t_0) = m, y'(t_0) = n, m, n \in R \tag{16}$$

will be written in the next form

$$y(t) = my_1(t) + \frac{1}{\beta(t_0)}[K(t_0)m + n]y_2(t) + y_3(t), t \in I, \tag{17}$$

where the functions $y_1(t), y_2(t)$ and $y_3(t)$ are defined by the formulas (5), (6) and (7).

Proof: Taking into account (5), (6), (7), (8), (10), (12) and (4) from (16) we obtain

$$c_1 = m, c_2 = \frac{1}{\beta(t_0)}[K(t_0)m + n]. \tag{18}$$

On the strength (18) we have the formula (17). Theorem 2 has been proved.

Example 1. We consider the equation (1) for

$$q(t) = K_0 p(t) - K_0^2 + a^2 exp\left\{4K_0 t - 2\int p(t)dt\right\}, t \in I, \tag{19}$$

where $K_0, a \in R, a \neq 0, p(t) \in C(I)$. In this case for $K(t) = K_0, \beta(t) = a\, exp\{2K_0 t - \int p(t)dt\}, t \in I,$ all conditions of theorem 1 hold. Therefore the common solution of the equation (1) will be written in the form (4), where

$$y_1(t) = exp[-K_0(t - t_0)]cos\left[\int_{t_0}^{t} \beta(s)ds\right],$$

$$y_2(t) = exp[-K_0(t - t_0)]sin\left[\int_{t_0}^{t} \beta(s)ds\right],$$

$$y_3(t) = \int_{t_0}^{t} exp\,[-K_0(t - s)]\frac{f(s)}{\beta(s)} sin\left[\int_{s}^{t} \beta(\tau)d\tau\right] ds, t_0, t \in I.$$

If $K_0 = 0, q(t) \in C^1(t), q(t) > 0$ for all $t \in I$, then from (19) we have

$$\beta(t) = \sqrt{q(t)}, p(t) = -\frac{q'(t)}{2q(t)}.$$

Then

$$y_1(t) = cos\left(\int_{t_0}^{t} \sqrt{q(s)}ds\right), y_2(t) = sin\left(\int_{t_0}^{t} \sqrt{q(s)}ds\right),$$

$$y_3(t) = \int_{t_0}^{t} \frac{f(s)}{\sqrt{q(s)}} sin\left[\int_{s}^{t} \sqrt{q(\tau)}d\tau\right]\, ds, t_0, t \in I.$$

If $q(t) = q_0 - const, q_0 > 0$, then $p(t) = 0, t \in I,$

$$y_1(t) = \cos\left[\sqrt{q_0}(t - t_0)\right], y_2(t) = \sin\left[\sqrt{q_0}(t - t_0)\right],$$

$$y_3(t) = \int_{t_0}^{t} \frac{f(s)}{\sqrt{q_0}} \sin\,[\sqrt{q_0}(t - s)]\, ds, t_0, t \in I.$$

Example 2: We consider the equation (1) for $p(t) \in C^1(I), f(t) \in C(I)$ and

$$q(t) = \frac{1}{4}p^2(t) + \beta_0^2 + \frac{1}{2}p'(t), t \in I,$$

where $\beta_0 \in R, \beta_0 \neq 0$. In this case formulas (2) and (3) hold for $K(t) = \frac{1}{2}p(t)$ and $\beta(t) = \beta_0, t \in I$. Then

$$y_1(t) = exp\left\{-\int_{t_0}^{t} \frac{1}{2} p(s)ds\right\} \cos[\beta_0(t - t_0)],$$

$$y_2(t) = exp\left\{-\int_{t_0}^{t} \frac{1}{2} p(s)ds\right\} \sin\,[\beta_0(t - t_0)],$$

$$y_3(t) = \int_{t_0}^{t} exp\left\{-\int_{s}^{t} \frac{1}{2} p(\tau)d\tau\right\} \frac{f(s)}{\beta_0} \sin[\beta_0(t - s)]\, ds.$$

If $\mathrm{p}(t) = p_0 - const, q(t) = \frac{1}{4}\, p_0^2 + \beta_0^2, \beta_0 \neq 0, t \in I$, then

$$y_1(t) = exp\left[-\frac{p_0}{2}(t - t_0)\right] \cos[\beta_0(t - t_0)],$$

$$y_2(t) = exp\left[-\frac{p_0}{2}(t - t_0)\right] \sin[\beta_0(t - t_0)],$$

$$y_3(t) = \int_{t_0}^{t} exp\left[-\frac{p_0}{2}(t - s)\right] \frac{f(s)}{\beta_0} \sin[\beta_0(t - s)]\, ds.$$

Example 3: We consider the equation (1) for $p(t) \in C^1(I), f(t) \in C(I), q(t) = \frac{1}{4}p^2(t) + +a^2t^{2\alpha} + \frac{1}{2}p'(t) - \frac{\alpha^2+2\alpha}{4t^2}, t \in I, I \subset (0, \infty), a, \propto \in R, a \neq 0, \propto \neq 0, \propto \neq -1$. Hence

$$q(t) = \frac{1}{4}\left[p(t) + \frac{\propto}{t}\right]^2 + a^2t^{2\alpha} + \frac{1}{2}\left[p'^{(t)} - \frac{\propto}{t^2}\right] - \frac{\propto}{2t}\left[p(t) + \frac{\propto}{t}\right], t \in I.$$

In this case the formulas (2) and (3) hold for $K(t) = \frac{1}{2}\left[p(t) + \frac{\propto}{t}\right], \beta(t) = at^{\alpha}, t \in I$.

Then

$$y_1(t) = exp\left\{-\int_{t_0}^{t} \frac{1}{2}\left[p(s) + \frac{\propto}{s}\right] ds\right\} \cos\left[\frac{\propto}{\propto + 1}(t^{\alpha+1} - t_0^{\propto+1})\right],$$

$$y_2(t) = exp\left\{-\int_{t_0}^{t} \frac{1}{2}\left[\left[p(s) + \frac{\propto}{s}\right]\right] ds\right\} sin\left[\frac{\propto}{\propto + 1}(t^{\alpha+1} - t_0^{\propto+1})\right],$$

$$y_3(t) = \int_{t_0}^{t} exp\left\{-\int_{s}^{t} \frac{1}{2}\left[p(\tau) + \frac{\propto}{\tau}\right] d\tau\right\} \frac{f(s)}{as^{\alpha}} \sin\left[\frac{\propto}{\propto + 1}(t^{\alpha+1} - s^{\alpha+1})\right] ds,$$

where $t_0, t \in I$.

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