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On M-Fuzzy Cosets, M-Conjugate of M-Upper Fuzzy Subgroups over M-Groups

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Abstract

This paper contains some definitions and results of the types of m- fuzzy cosets of a m- groups and m – conjugate upper fuzzy subgroups of a m – groups. Using m – upper normal fuzzy subgroups, m – conjugate upper fuzzy subgroups and upper fuzzy level subset in the types of m –fuzzy coset of a m – groups is studied. Some properties and corollaries are also established and discussed for this subject.

Keywords: Fuzzy Sets; Positive Fuzzy Subsets; Upper Fuzzy Level Subset; M-Upper Fuzzy Subgroups; M – Upper Normal Fuzzy Subgroups; M – Fuzzy Coset; M- Positive Fuzzy Coset; M- Double Fuzzy Coset; M – Positive Double Fuzzy Coset; M- Conjugate Upper Fuzzy Subgroups.

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Introduction

The concept of fuzzy sets was initiated by Zadeh [16]. Then it has become a vigorous area of research in analysis, topology, probability, statistic and measure theory etc. In [13] Rosenfeld used this concept to develop the theory of fuzzy groups. In fact, many basic properties in group theory are found to be carried over to fuzzy groups. Aktas and Cagman [1] studied the product of fuzzy subgroups and t – level subgroup. [2] Aktas discussed fuzzy quotient groups. Several mathematicians have followed the Rosenfeld approach in investigating the fuzzy subgroup theory. Fuzzy normal subgroups were studied by Wu [15] Kumar [7]. The concept of upper Q- fuzzy index order with upper Q – fuzzy subgroups studied by Nagarajan and Solairajn [4,9] a fuzzy subgroups of finite abelian groups, fuzzy subgroups of finite cyclic groups and fuzzy subgroup of symmetric groups S_2, S_3 and alternating group A_4 was studied by some authors [11,14,12], N.Jacobson [5] introduced the concept of m – groups, m – subgroups. Some authors [3, 6, 8, 10] discussed the homomorphism and anti – homomorphism of fuzzy subgroups and give some properties for this subject.

In this work, we first generalize the definition of fuzzy cosets and its types over group to fuzzy cosets over m – groups by using the definition of m – upper fuzzy subgroups. Then we study some of their properties.

Preliminary

Definition2.1 [8] Let G be a group, M be a set, if

1. $mx \in G \; \forall \; x \in G , m \in M$
2. $m (xy) = (mx)y = x (my) \; \forall \; x , y \in G , m \in M$. Then m is said to be a left operator of G, M is said to be a left operator set of G. G is said to be a group with operators. We use phrase "G is an m – group" in stead of a group with operators. If a subgroup of m – group G is also M-group, then it is said to be an m – subgroup of G.

Defintion2.2 Let μ be a fuzzy subset of X. for $t \in [0 , 1]$ the upper level subset of μ is the set $\mu_t = \{ x \in X , \mu (x) \leq t \}$.this is called an upper fuzzy level subset of μ.

Definition2.3 [16] Let X be a non empty set, a fuzzy set μ is just a function from X onto [0 , 1].

Definition2.4 [16] A fuzzy set μ on X is called positive fuzzy set if

$\mu (x) > 0 \; \forall \; x \in X$.

Definition2.5 Let G be a m – group. A fuzzy subset of G is said to be an m – upper fuzzy subgroup of G if it satisfies the following conditions:-

1. $\mu (mxy) \leq \max \{\mu (x) , \mu (y)\}$
2. $\mu (x^{-1}) \leq \mu (x) ; \forall \; x , y \in G$.

Definition2.6 Let G be a m – group. A fuzzy subset μ of G is said to be m – upper fuzzy subgroup of G if $\mu (mxy^{-1}) \leq \max \{\mu (x) , \mu (y)\}$.

For all $x , y \in G$.

Definition2.7 Let μ be an m – upper fuzzy subgroup of a m – group G and $H = \{ x \in G ; \mu (x) = \mu (e) \}$. Then O (μ) order of μ is defined as O (μ) = O (H).

Proposition2.8 $H = \{x \in G ; \mu (x) = \mu (e) \}$ is m – subgroup when μ is an m – upper fuzzy subgroup.

Definition2.9 Let G be a m – Group: An m – upper fuzzy

Subgroup μ of G is said to be m – upper normal fuzzy subgroup of G if μ (xy)

$= \mu (yx)$ for all $x, y \in G$.

M – Conjugate and Types of M – Fuzzy Cosets

Definition3.1 Let μ, δ be two m –upper fuzzy subgroups of m – group G. Then μ, δ are said to be m – conjugate upper fuzzy subgroups of G if for some $g \in G$,

$\mu (x) = \delta (g^{-1}x g)$ for every $x \in G$.

Proposition3.2 If μ, δ are two fuzzy subset of abelian m –group G, then μ, δ are conjugate fuzzy subsets of G iff $\mu = \delta$.

Proof: Straightforward.

Proposition3.3 If μ, δ are m – conjugate upper fuzzy subgroups of m – group G, then $O (\mu) = O (\delta)$.

Proof: Straightforward.

Theorem3.4 Let μ be m – upper fuzzy subgroup of a m – group G and δ be a fuzzy subset of G, if μ and δ are conjugate fuzzy subset of G. Then δ is m – conjugate upper fuzzy subgroup of a m – group G.

Proof

Let μ be m – upper fuzzy subgroup of a m – group G and δ be a fuzzy subset of G and let μ and δ are conjugate fuzzy subset of G, we need to prove δ is m – conjugate upper fuzzy subgroup of a m – group G. Let $x, y \in G$ and $m \in M$, then $mxy^{-1} \in G$

$$\begin{aligned} \delta (mxy^{-1}) &= \mu (mg^{-1}xy^{-1}g) \text{ for some } g \in G \\ &= \mu (mg^{-1}xgg^{-1}y^{-1}g) \\ &= \mu((mg^{-1}xg) (g^{-1}y^{-1}g)) \\ &= \mu((mg^{-1}xg) (g^{-1}yg)^{-1}) \\ &\leq \max\{\mu (g^{-1}xg), \mu (g^{-1}yg)^{-1}\} \\ &\leq \max\{\mu (g^{-1}xg), \mu (g^{-1}yg)\} \end{aligned}$$

Since μ is m – upper fuzzy subgroup of G.

$\leq \max \{\delta (x) , \delta (y) \}$

μ and δ are conjugate fuzzy subset of G, therefore $\delta (mxy^{-1}) \leq$

$\max \{\delta (x) , \delta (y) \}$. Hence δ is m – conjugate upper fuzzy subgroup of a m – group G.

Definition 3.5 Let μ is an m – upper fuzzy subgroup of a m – group G. For any $a \in G$,

$a\mu$ defined by $(a\mu)(x) = \mu(a^{-1}x)$ for every $x \in G$ is called m – fuzzy coset of the m – group G.

Lemma3.6 If μ is m – upper fuzzy subgroup of m – group G, then
$\mu(e) \leq \mu(x)$ for every $x \in G$.

Proposition3.7 If μ is m – upper normal fuzzy subgroup of a m – group G, then the set $G/\mu = \{a\mu ; a \in G\}$ is m – group with operation $(a\mu)(b\mu) = (ab)\mu$.

Proof
Let $a, b \in G$, $a\mu, b\mu \in$
G/μ. It's clear that $b^{-1} \in G$ thus
$b^{-1}\mu \in G/\mu$, $(a\mu)(b^{-1}\mu) =$
$(ab^{-1}) \in G/\mu$. Therefore G/μ is a m – group, we get $(G/\mu, .)$ is called the m – fuzzy factor group of G with μ.

Theorem3.8 Let μ be m – upper fuzzy subgroup of a m – group G. Then $a\mu = b\mu$ for any $a, b \in G$ iff
$\mu(a^{-1}b) = \mu(b^{-1}a) = \mu(e)$.

Proof
$\Rightarrow$ Let μ be m – upper fuzzy subgroup of m – group G, let $a\mu = b\mu$ for any $a, b \in G$. Then $(a\mu)(a) = (b\mu)(a)$ and $(a\mu)(b) = (b\mu)(b)$
Thus $\mu(a^{-1}a) = \mu(b^{-1}a)$ and $\mu(a^{-1}b) = \mu(b^{-1}b)$, hence $\mu(e) = \mu(b^{-1}a)$ and $\mu(a^{-1}b) = \mu(e)$. Therefore $\mu(a^{-1}b) = \mu(b^{-1}a) = \mu(e)$.
$\Leftarrow$ Suppose $\mu(a^{-1}b) = \mu(b^{-1}a) = \mu(e)$ for any $a, b \in G$, for every $g \in G$ and we have

$$\begin{aligned} a\mu(g) &= \mu(a^{-1}g) \\ &= \mu(a^{-1}bb^{-1}g) \\ &\leq \max\{\mu(a^{-1}b), \mu(b^{-1}g)\} \\ &\leq \max\{\mu(e), \mu(b^{-1}g)\} \\ &= \mu(b^{-1}g) \\ &= b\mu(g). \end{aligned}$$

Thus $a\mu(g) \leq b\mu(g)$….(i)
On the other hand

$$b\mu(g) = \mu(b^{-1}g)$$

$= \mu (b^{-1}aa^{-1}g)$

$\leq \max \{\mu (b^{-1}a) , \mu (a^{-1}g)\}$

$\leq \max \{ \mu (e) , \mu (a^{-1}g)\}$

$= \mu (a^{-1}g)$

$= a\mu (g)$.

Thus $b\mu (g) \leq a\mu (g)$….(ii)

By (i) and (ii) we get $a\mu (g) =$

$b\mu (g)$, hence $a\mu = b\mu$.

Theorem3.9 Let μ be m – upper fuzzy subgroup of a m – group G and $a\mu = b\mu$ for a, $b \in G$. Then

$\mu (a) = \mu (b)$.

Proof: Straightforward.

Theorem3.10 Let μ be m – upper fuzzy subgroup of a m – group G. Then $a \mu_t = b\mu_t$ for any a, $b \in G- \mu_t$

Then $\mu (a) = \mu (b)$.

Proof

Let μ be m – upper fuzzy subgroup of m – group G, let $a \mu_t = b\mu_t$ for any a, $b \in G - \mu_t$, $t \in$

[0 , 1] but $b^{-1}a$ and $a^{-1}b \in \mu_t$

$\mu (a) \quad = \mu (bb^{-1}a)$

$\leq \max \{ \mu (b) , \mu (b^{-1}a) \}$

$= \max \{ \mu (b) , \mu (e) \}$

$= \mu (b)$, by Lemma 3.6

Thus $\mu (a) \leq \mu (b)$ and

$\mu (b) \quad = \mu (aa^{-1}b)$

$\leq \max \{ \mu (a) , \mu (a^{-1}b) \}$

$= \max \{ \mu (a) , \mu (e) \}$

$= \mu (a)$, by Lemma 3.6

Thus $\mu (b) \leq \mu (a)$

We get $\mu(a) = \mu(b)$.

Theorem3.11 Let μ be m – upper fuzzy subgroup of a m – group G. Then a $\mu_t = (a\mu)_t$ for every $a \in G$ and $t \in [0, 1]$.

Proof
Let μ be m – upper fuzzy subgroup of m – group G and let $x \in G$.

$$x \in (a\mu)_t \Leftrightarrow (a\mu)(x) \le t$$
$$\Leftrightarrow \mu(a^{-1}x) \le t$$
$$\Leftrightarrow a^{-1}x \in \mu_t$$
$$\Leftrightarrow x \in a\mu_t$$

Thus $a\mu_t = (a\mu)_t$ for every $x \in G$.

Definition3.12 Let μ be m – upper fuzzy subgroup of a m – group G, $a \in G$ and λ be positive fuzzy set. Then the m – positive fuzzy coset $(a\mu)^{\lambda}$ is defined by $(a\mu)^{\lambda}(x) = \lambda(a)\mu(x)$ for every $x \in G$.

Theorem3.13 Let μ be m – upper fuzzy subgroup of a m – group G, then the m – positive fuzzy coset $(a\mu)^{\lambda}$ is m – upper fuzzy subgroup of a m – group G, for every $a \in G$.

Proof
Let μ be m – upper fuzzy subgroup of m – group G, for every $x, y \in G$ we have

$$(a\mu)^{\lambda}(mxy^{-1}) = \lambda(a)\,\mu(mxy^{-1})$$
$$\le \lambda(a) \max\{\mu(x), \mu(y)\}$$
$$= \max\{\lambda(a)\,\mu(x), \lambda(a)\,\mu(y)\}$$
$$= \max\{(a\mu)^{\lambda}(x), (a\mu)^{\lambda}(y)\}$$

Thus $(a\mu)^{\lambda}(mxy^{-1}) \le \max\{(a\mu)^{\lambda}(x), (a\mu)^{\lambda}(y)\}$, therefore $(a\mu)^{\lambda}$ is m – upper fuzzy subgroup of a m – group G.

Corollary3.14 Let λ be positive fuzzy set and μ be m – upper normal fuzzy subgroup of a m – group G, then the m – positive fuzzy coset $(a\mu)^{\lambda}$ is m – upper normal fuzzy subgroup of a m – group G, for every $a \in G$.

Corollary3.15 Let μ,δ be m – upper fuzzy subgroups of a m – group G and λ be positive fuzzy set. Then for all $a \in G$

$(a\mu)^{\lambda} \subseteq (a\delta)^{\lambda}$ iff $\mu \subseteq \delta$.

$(a\mu)^{\lambda} = (a\delta)^{\lambda}$ iff $\mu = \delta$.

Definition3.16 Let μ be m – upper fuzzy subgroup of a m – group G. Then for a, b ∈ G the m – double fuzzy coset aμb of G is defined by

$(a\mu b)(x) = \mu(a^{-1}xb^{-1})$ for every x ∈ G.

Theorem3.17 If μ is an m – upper fuzzy subgroup of a m – group G. Then for any a∈ G the m – double fuzzy coset $a\mu a^{-1}$ of G is also m – upper fuzzy subgroup of G.

Proof

Let μ be m – upper fuzzy subgroup of m – group G and a ∈ G, let x ,y ∈ G. Then

$$(a\mu a^{-1})(mxy^{-1}) = \mu(ma^{-1}xy^{-1}a)$$
$$= \mu(ma^{-1}xaa^{-1}y^{-1}a)$$
$$= \mu(m(a^{-1}xa)(a^{-1}y^{-1}a))$$
$$\le \max\{\mu(a^{-1}xa), \mu(a^{-1}ya)^{-1}\}$$
$$\le \max\{\mu(a^{-1}xa), \mu(a^{-1}ya)\}$$

Since μ is m – upper fuzzy subgroup, then

$$= \max\{(a\mu a^{-1})(x), (a\mu a^{-1})(y)\}$$

Thus

$(a\mu a^{-1})(mxy^{-1}) \le \max\{(a\mu a^{-1})(x), (a\mu a^{-1})(y)\}$.

Therefore $a\mu a^{-1}$ is m – upper fuzzy subgroup of a m – group G.

Corollary3.18 If μ is m – upper normal fuzzy subgroup of a m – group G. Then for any a ∈G, the m – double fuzzy coset $a\mu a^{-1}$ of G is also m – upper normal fuzzy subgroup of G.

Theorem3.19 Let μ be m – upper fuzzy subgroup of a m – group G and $a\mu a^{-1}$ be m – double fuzzy coset of G. Then $O(a\mu a^{-1}) = O(\mu)$ for any a ∈ G.

Proof

Let μ be m – upper fuzzy subgroup of m – group G and a ∈ G. By Theorem 3.17, the m – double fuzzy coset $a\mu a^{-1}$ is m – upper fuzzy subgroup of G also by the Definition 3.16 we have $(a\mu a^{-1})(x) = \mu(a^{-1}xa)$, for every x ∈ G. Then for any a ∈ G , μ and $a\mu a^{-1}$ are m – conjugate upper fuzzy subgroup of m – group G as there exist a∈ G such that $(a\mu a^{-1})(x) = \mu(a^{-1}xa)$ for every x by Proposition3.3 $O(a\mu a^{-1}) = O(\mu)$ for any a ∈ G.

Definition3.20 Let μ, δ be two m – upper fuzzy subgroups of m – group G. And λ be a positive fuzzy set, then for a ∈G we define the m – positive double fuzzy coset

$(\mu\, a\, \delta)^{\lambda}$ by :-

$(\mu\, a\, \delta)^{\lambda} = \max\{(a\mu)^{\lambda}, (a\delta)^{\lambda}\}$

Theorem3.21 The m – positive double fuzzy coset $(\mu\, a\, \delta)^{\lambda}$ is m – upper fuzzy subgroup of m – group G, when μ , δ are m – upper fuzzy subgroups of G.

Proof

For every x , y ∈G, we have

$(\mu\, a\, \delta)^{\lambda}(mxy^{-1})$

$=\max\{(a\mu)^{\lambda}(mxy^{-1}),(a\delta)^{\lambda}(mxy^{-1})\}$

$=\max\{\lambda(a)\mu(mxy^{-1}),\lambda(a)\delta(mxy^{-1})\}$

$\leq\lambda(a)\max\{\max\{\mu(x),\mu(y)\}, \max\{\delta(x),\delta(y)\}\}$

$=\lambda(a)\max\{\max\{\mu(x),\delta(x),\mu(y),\delta(y)\}$

$=\lambda(a)\max\{\max\{\mu(x),\delta(x)\}, \max\{\mu(y),\delta(y)\}\}$

$=\max\{\lambda(a)\max\{\mu(x),\delta(x)\},$

$\lambda(a) \max\{\mu(y),\delta(y)\}\}$

$= \max\{(\mu\, a\, \delta)^{\lambda}(x), (\mu\, a\, \delta)^{\lambda}(y)\}$

Thus $(\mu\, a\, \delta)^{\lambda}(mxy^{-1}) \leq$

$\max\{(\mu\, a\, \delta)^{\lambda}(x), (\mu\, a\, \delta)^{\lambda}(y)\}$

Hence $(\mu\, a\, \delta)^{\lambda}$ is m – upper fuzzy subgroup of m – group G.

Corollary3.22 If μ, δ are m – upper normal fuzzy subgroups of m – groups G. Then the m – positive double fuzzy coset $(\mu a \delta)^{\lambda}$ is m – upper normal fuzzy subgroup of G.

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