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Quantization of Charge in the Standard Model

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Abstract

The quantization of the gauge charge in the standard model is derived from a compatibility condition for Abelian bundles over manifolds occuring in the geometric description of the internal symmetry spaces. The quantization of the electric charge and other Abelian charges of composite hadron states may be found by considering a flux quantization condition which is required for a global Lagrangian formulated on a U(1) bundle over the initial base space. The values of the quark charges are determined by the charges of the composite states in isospin multiplets.

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1. Introduction

The confinement of the fractional charge of the quark is consistent with the introduction of a linearly rising potential generated by the exchange of gluons. As the separation between the quark and anti-quark increases, there is sufficient energy in the vacuum to create another quark- anti-quark pair. It is known also that quantum chromodynamics has the property of asymptotic freedom, implying that the quarks propagate almost freely at very short distances.

Support for the confinement of quarks arises from the proportionality of the Wilson loop operator for a quark-anti-quark pair with the area. Furthermore, no fractional charge has been observed independently in the final state of a reaction. However, the analytic proof of the existence of a potential, derived from a force field being governed by strings, is complicated by the perturbative evaluation of its path integral representation.

Nevertheless, a theoretical explanation can be provided for the values of the fractional charges of the quarks in the standard model, given the charges of composite states in isospin multiplets. It might be anticipated that these techniques could be useful in an investigation of quark confinement. The convergence of the perturbation series about nonperturbative states such as quark-anti-quark pairs and instanton-anti-instanton configurations [1], and the divergence of the expansion for values of the isospin less than a fixed bound [2], confirms that there is a effective description of the strong interactions in terms of the composite states which provides a basis for the non-observability of free quarks.

2. Quantization of Charge and Isospin in the Standard Model

The existence of only a local Lagrangian can be used to construct a global Lagrangian with the augmentation of the total space by a $U(1)$ bundle. Suppose that the curvature form Ω on a space is closed but not exact. An exact form $\tilde{\Omega}$ may be defined on the $U(1)$ bundle over this space such that $\tilde{\Omega} = d\Lambda$ where Λ is a global one-form and the projected form is Ω. It has been demonstrated that if there is a local Lagrangian with a global Hamiltonian H and symplectic form ω such that the integral of ω over any closed two-surface in the configuration space is quantized, then there exists a $U(1)$ bundle admitting a global Lagrangian [3].

These results may be used as a basis for the quantization of charge in the standard model. While the Lagrangian of quantum electrodynamics is globally defined on four-dimensional Minkowski space-time, it must be combined with the weak and strong nuclear forces in a complete description of the elementary particle interactions. In the $SU(3)_c \times SU(2)_L \times U(1)_Y$ standard model, the $U(1)$ electric charge is derived from a linear combination of the hypercharge Y and the isospin I_3. It may be recalled that the Hopf fibration $S^3 \overset{U(1)}{\rightarrow} S^2$ is nontrivial and the $U(1)$ action is not globally defined on $S^2 \simeq SU(2)/U(1)$. Instead, it is given on two overlapping patches, each of which can be retracted to a hemisphere, and thereby, the conditions for the previously stated theorem hold. Since the $U(1)$-invariant Lagrangian of quantum electrodynamics must include this $U(1)$ action on $SU(2)_L$, it is only locally defined on the space $(SU(3) \times SU(2))/(SU(2) \times U(1)') \times M^4$. Likewise, the $U(1) \times U(1)$ bundle over $(SU(3) \times SU(2))/(SU(2) \times U(1)' \times U(1)'') \times M_4$ might be used to deduce the quantization of two U(1) charges. Consequently, quantization of the electric charge and other quantum numbers would be expected in the standard model.

This method then can be adapted to define quantized Abelian charges in the standard model through compatibility of the transition functions on the intersection of overlapping neighbourhoods on the base space of the bundle. For a U(1) bundle, the compatibility conditions have the form

$$g_{\alpha\beta} = e^{iqf_{\alpha\beta}} \tag{2.1}$$
$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$$

where the transition functions in the overlap of the two open sets U_α and U_β take values in U(1). Since a U(1) gauge transformation has the form $g = e^{iq\chi}$, such that the potential transforms as $A_\mu \to A_\mu + q\partial_\mu \chi$, the quantization of q follows from the condition

$$q(f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\delta}) = 2\pi n, \qquad n \in \mathbb{Z}. \tag{2.2}$$

When a similar quantization condition is placed on the exponential factor $e^{iq\int A_\mu dx^\mu}$, and the different components of the line integral over a contour in a four-dimensional region with the angular variables taking values in half of the interval $[0, 2\pi]$, without the imaginary unit, are identified with the electric charge, the negative of the baryon number and the negative of the strangeness, quantization of each of these Abelian charges for the composite hadron states can be derived [4].

While the Wilson loop factor initially differs from the integral over the curvature form, these may be related by Stokes' theorem since

$$\int_S \tilde{\Omega} = \int_S d\Lambda = \int_C \Lambda \tag{2.3}$$

where the one-form Λ can be identified with $iqA_\mu dx^\mu$.

3. The Quark Charges

The octupole moment of vibrations of the nucleus has angular momentum 3. An analogous method might be used to deduce the fractional charge of quarks in the proton and neutron. The charge density of a nucleon can be derived from the form factor

$$F(\vec{q}) = \int \rho(\vec{r}) e^{-i\vec{q}\cdot\vec{r}} d^3\vec{r}. \tag{3.1}$$

As

$$e^{-i\vec{q}\cdot\vec{r}} = 1 - i\vec{q}\cdot\vec{r} - \frac{(\vec{q}\cdot\vec{r})^2}{2!} + i\frac{(\vec{q}\cdot\vec{r})^3}{3!} - \ldots, \tag{3.2}$$

$i\int \rho(\vec{r})\frac{(\vec{q}\cdot\vec{r})^3}{3!}d^3\vec{r}$ would be the octupole moment. By identifying the axes in the diagram with the coordinates of the octonions, a link is established between this algebra and the presence of substructure of nucleons consisting of three components.

The octonionic module has been found to be necessary for the inclusion of quarks in the spinor space defined by a tensor product of division algebras. The octonions furthermore can be viewed as a semidirect product of three copies of $\mathbb{C}$. It is known also that SU(3) is an appropriate symmetry group for strongly interacting fermions while the projection from S^7 to $S^3 \times S^3$ is useful for a theoretical explanation of the properties of the vector bosons. Then the existence of three fractional charges adding to the charges of the proton and neutron, with these two particles forming an isospin doublet, is constrained by the relations

$$\begin{aligned} q_1 + q_2 + q_3 &= 1 \\ q_1' + q_2' + q_3' &= 0 \end{aligned} \tag{3.3}$$

where the triplet $\{q_1', q_2', q_3'\}$ can be formed from the triplet $\{q_1, q_2, q_3\}$ after the interchange of one charge. There are three possible cases.

I. $q_1 = q_2 = q_3$

Since $q_1 + q_2 + q_3 = 1$, $q_1 = q_2 = q_3 = \frac{1}{3}$. An interchange of any two quarks yields a triplet with the same charge $\frac{1}{3}$, and thus, $q_1' + q_2' + q_3' \neq 0$.

II. $q_1 = q_2 \neq q_3$

Then

$$\begin{aligned} 2q_1 + q_3 &= 1 \\ q_3 &= 1 - 2q_1 \end{aligned} \tag{3.4}$$

The interchange yields a triplet with charges $q_1' = q_1$, $q_2' = q_3' = q_3$. Then

$$q_1' + q_2' + q_3' = q_1 + 2q_3 = 0 \tag{3.5}$$

and

$$\begin{aligned} q_3 &= -\frac{1}{2}q_1 \\ -\frac{1}{2}q_1 &= 1 - 2q_1 \\ \frac{3}{2}q_1 &= 1 \\ q_1 &= \frac{2}{3} \\ q_3 &= 1 - 2\left(\frac{2}{3}\right) = -\frac{1}{3} \end{aligned} \tag{3.6}$$

III. $q_1 \neq q_2 \neq q_3$

As $q_3 = 1 - q_1 - q_2$, an interchange of the charges does not change the triplet $\{q_1', q_2', q_3'\}$. Therefore, the properties of the isospin operator must be used.

If

$$\begin{aligned} &q_1' = q_1 - 1 \quad && q_2' = q_2 \quad && q_3' = q_3 \\ &q_1' = q_1 && q_2' = q_2 - 1 && q_3' = q_3 \\ &q_1' = q_1 && q_2' = q_2 && q_3' = q_3 - 1 \end{aligned} \tag{3.7}$$

it follows that

$$q_1' + q_2' + q_3' = (q_1 + q_2 + q_3) - 1 = 0 \tag{3.8}$$

However, $\{q_1', q_2', q_3'\}$ also must equal q_2 or q_3. If $q_1 - 1 = q_2$,

$$\begin{aligned} 1 - q_1 - q_2 &= 1 - q_1 - (q_1 - 1) = 2(1 - q_1) \\ q_1 + q_2 + q_3 &= q_1 + q_1 - 1 + 2(1 - q_1) = 1 \end{aligned} \tag{3.9}$$

The latter triplet has the property $q_1' = q_2' \neq q_3'$ and therefore it belongs to class II. Inverting the roles of $\{q_1, q_2, q_3\}$ and $\{q_1', q_2', q_3'\}$ implies that $\{q_1, q_2, q_3\}$ should consist of only two values of charges. Since this conclusion is contrary to the characteristic of class III triplets, the charges must be

$$\{q_1, q_2, q_3\} = \left\{\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\} \qquad (3.10)$$
$$\{q_1', q_2', q_3'\} = \left\{\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right\}$$

which is consistent with the known values for the u and d quarks.

4. Extension of the Quantization to Fractional Values

The theorem in §2 provides a theoretical basis for the quantization of charge in the standard model. However, the relative fractional value of the quark to lepton electric charges is left undetermined. A further examination of the bundle structure shows that the Lagrangian describing a unification of the electromagnetic, weak and strong nuclear forces is defined locally on $(SU(3) \times U(1))/(SU(2) \times U(1)') \times M_4$ and globally on an $SU(2)$ bundle over this manifold or locally on $(SU(2) \times U(1))/(SU(2) \times U(1)') \times M_4$ and globally on an $SU(3)$ bundle over this base space.

The extension of the argument for the quantization of electric charge could be considered for these bundles. Instead of the curvature form equalling $d\theta_\alpha$ on a neighbourhood U_α, such that $d(\theta_\alpha - \theta_\beta) = 0$ on $U_\alpha \cap U_\beta$ or $\theta_\alpha - \theta_\beta = df_{\alpha\beta}$, such that

$$d(f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}) = 0 \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \qquad (4.1)$$

and

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 2\pi n_{\alpha\beta\gamma}\lambda \qquad n \in \mathbb{Z} \qquad (4.2)$$

with the transition functions

$$g_{\alpha\beta} = e^{\frac{if_{\alpha\beta}}{\lambda}} \qquad (4.3)$$

satisfying

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \qquad (4.4)$$

the replacement $\lambda \to \bar{\chi}\vec{\sigma} \cdot \vec{\tau}\chi$ for $SU(2)$ yields relations of the form

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 2\pi n_{\alpha\beta\gamma}\bar{\chi}\vec{\sigma} \cdot \vec{\tau}\chi \qquad (4.5)$$

and

$$g_{\alpha\beta} = e^{\frac{if_{\alpha\beta}}{\bar{\chi}\vec{\sigma}\cdot\vec{\tau}\chi}} \qquad (4.6)$$

However, if $\chi_1 = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$, the bilinear equals

$$(\chi_1^* \chi_2 + \chi_2^* \chi_1)\tau_1 + i(-\chi_1^* \chi_2 + \chi_2^* \chi_1)\tau_2 + (\chi_1^* \chi_1 - \chi_2^* \chi_2)\tau_3 \tag{4.7}$$

this form is too complex for the description of the group element. Moreover, it must be contained in a product of $U(1)$ factor, which is indicative of the embedding of $U(1)$ into the larger group $SU(2)$ rather than the entire nonabelian group. A similar conclusion would hold for $SU(3)$.

The diagonal embeddings of $U(1) \times ... \times U(1)$ into $U(n)$ would consist of matrices of the form

$$\begin{pmatrix} e^{2\pi i \alpha_1} & ... & 0 \\ 0 & ... & 0 \\ 0 & ... & e^{2\pi i \alpha_n} \end{pmatrix} \tag{4.8}$$

The restriction to $SU(n)$ would imply that $\alpha_1 + ... + \alpha_n = k$ for some integer k.

The most trivial constraint would be

$$\alpha_1 + ... + \alpha_n = 0 \tag{4.9}$$

which, for $n = 2$, is equivalent to $\alpha_1 = -\alpha_2$. A constraint of this form would imply the existence of charges of opposite sign. When $k = 1$, the constraint

$$\alpha_1 + ... + \alpha_n = 1 \tag{4.10}$$

would be satisfied by fractional values of the parameters α_i. A symmetric embedding, with $\alpha_1 = ... = \alpha_n$ would imply that $\alpha_i = \frac{1}{n}$ for all n. When $n = 2$, $\alpha_i = \frac{1}{2}$, whereas, if $n = 3$, it follows that $\alpha_i = \frac{1}{3}$. For a less symmetric distribution of exponents, in $SU(2)$, the values for a doublet could be given by $\{0, 1\}$, while the values for $SU(3)$ could be $\left\{\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\}$.

When $k \geq 2$, the values of α_i can be chosen to multiples of those for $k = 1$. This would imply that the basic units of charge would be determined by the values for $k = 1$.

It may be recalled that Eq.(4.2) led to the quantization of the two-surface integral

$$\int_\Sigma \Omega = 2\pi \nu \lambda \qquad \nu = 0, \pm 1, \pm 2, ... \tag{4.11}$$

The integral values of ν may be replaced by the fractional values for the set $\{\alpha_i\}$.

Presuming that the integral of the curvature over the $SU(2)$ and $SU(3)$ bundles does not show any discontinuities, the symmetric diagonal embedding is preferable. Consequently, the integrals would have the form

$$\int_\Sigma \Omega = 2\pi \nu_{SU(2)} \lambda \qquad \nu_{SU(2)} = 0, \pm\frac{1}{2}, \pm 1, ... \tag{4.12}$$

and

$$\int_{\Sigma} \Omega = 2\pi \nu_{SU(3)}\lambda \qquad \nu_{SU(3)} = 0, \pm\frac{1}{3}, \pm\frac{2}{3}, ... \tag{4.13}$$

The electric charges of leptons in $SU(2)$ doublets are known to be integer. It may be surmised, therefore, that the $U(1)$ charge being quantized in Eq.(4.12) is different from the electric charge or the hypercharge. However, the third component of the isospin satisfying the Gell-Mann-Nishijima formula

$$Q = I_z + \frac{1}{2}Y \tag{4.14}$$

can have half-integral values, which are consistent with the quantum numbers of the leptons.

Finally, Eq.(4.13) confirms the quantization of the electric charge of the quarks as multiples of $\frac{1}{3}$ of the electron charge. The method of §3 then can be used to establish the values for each of the quarks in the three generations.

5. Conclusion

The values of the charges of the quarks in the standard model can be determined from group theoretical considerations for baryons belonging to isospin multiplets. The quantization of the charge also follows from the consistency condition for the overlap of the curvature form of a nontrivial bundle representing the gauge invariances of the standard model. The electric charges are identified through an embedding of $U(1) \times U(1) \times U(1)$ in $SU(3)$. The Gell-Mann-Nishijima formula is related to the quantization condition for the definition of a global Lagrangian. The flux integral may be shown to be equivalent to the two-dimensional integral of the curvature form of the U(1) field which represents the connection of the U(1) bundle required for a global formulation of the Abelian symmetries over in the higher-dimensional description of the standard model.

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