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Optimal Portfolio Selection for Pension Funds with Variable Rate of Return and Transaction Costs: Finite Horizon Case.

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Abstract

This paper investigates a Constant Relative Risk Aversion (CRRA) investor as a representative of pension plan participants, who has a finite investment horizon and is subject to the proportional transaction costs and a rate of return that is a linear function of time. We attempt to maximize the investor's utility by trading between stock and money market account. A set of partial differential equations are derived and closed form solution proffered. The effects of the volatility of the risky asset are investigated and the results show that all the parameters are horizon-dependent. A zero value of the volatility resulted to the value function equals zero and its unit value with the drift parameter $\xi(t)$ equals the discount function k, $(\xi(t) = k)$ led to the value function becoming indeterminate. The pension fund considers the horizon dependency when making policy for her client. Precise conditions are obtained which determine the growth rate of the value function in the sell and buy regions.

Keywords: optimal portfolio selection, pension funds, variable rate of return, transaction costs, CRRA.

Subject classification: 91B80, 91G30.

Introduction
Traditional economic models on optimal investment and consumption policy have been extended on many directions ever since Merton (1971, 1973) published his

pioneer article. In these benchmark models investors can trade asset continuously at any time without incurring any kind of costs. However, in the capital market, an asset is also featured by its liquidity in addition to the commonly used risk and return. Trading is most of the time costly.

Working with models that incorporate liquidity considerations, in whatever shape, requires a thorough re-examination of mainstream theory of financial markets. There are various ways to model illiquidity in an optimal investment problem. In this paper, transaction cost is used as a proxy for illiquidity due to the following two advantages: first it allows for mathematical flexibility and tractability; secondly it enables references and comparisons with a large body of previous literature. Amihud and Mendelson (1986), Acharya and Pederson (2005) among others derived clientele effect and spread-return in asset pricing with bid-spread ask and a liquidity-adjusted CAPM respectively. Longstaff (199) ; Schwartz and Tebaldi (2004), Davis and Norman (1990), considered an infinite horizon maximization problem with intermediate consumption, while Chellathurai and Draviam (2007) considered a finite horizon portfolio selection without intermediate consumption when fixed and/or proportional transaction costs are present. An efficient and tractable numerical algorithm to obtain the boundaries using binomial approximation was done by Gennotte and Jung (1992) .Zeriphopoulou et al (1993) solved the European option pricing problem with transaction costs taking advantage of convergence of discretization of the stock price. Similarly, Balduzzi and Lynch (1999) discrete both time and state to numerically compute the optimal investment policy for an investor with finite horizon. Jang et al (2007) found that in contrast to the standard literature, transaction costs can have a first order effect, and investor responds to changes in either regime by adjusting consumption and investment policies if the market conditions change over time.

On the other hand, Constantinides (1979, 1986) has shown that the optimal transaction policy is to maintain the ratio of the dollar amount invested in the riskless asset to that in the risky asset within a certain range, represented by the buy boundary and sell boundary. Consequently three regions are identified depending on the portfolio ratio: the no-transaction region, the buy region and the sell region.

A continuous-time dynamic maximization problem is formulated herein and the optimal conditions derived. In particular, a CRRA investor investigated as a representative of pension plan participants, who has a finite horizon and is subject to proportional transaction costs when trading stock as well as money market account. Mathematically these conditions are boiled down to a set of parabolic differential equations. In contrast to the infinite horizon model in which stationary solutions can be obtained, the value function and the corresponding two boundaries strongly depend on horizon. Liu and Loewinstein (2002) solved the deterministic finite horizon problem by making use of the exponentially distributed horizon. They claimed that proportional transaction costs together with a finite horizon would imply a time-varying, largely buy-and-hold trading strategy.

In our case, a closed form solution of the partial differential equation is given .It is found that all the parameters are horizon-dependent .The effects of the volatility of the risky asset is investigated. We show that a zero value of the volatility resulted to the value function equals zero and its unity value with $\xi(t) = k$ led to the value function becoming indeterminate.

The Models

The case of no-transaction costs

We shall assume that pension funds can trade two assets continuously in an economy. The first asset is the money market account (BOND) growing at a rate r_t that is linear function of time $(r_t = \alpha + \sigma t)$, instead of a constant as in Miao (2010) . $r_t = \alpha + \sigma t, (\alpha > 0, 0 \leq \sigma \leq 1)$ is a decreasing (or an increasing) linear function of $t \; as \; t \to \infty$ .The parameter α is the initial investment on the money market account which determines the speed of a mean-reversion to the stationary level. σ is the acceleration coefficient which is the volatility (variance) of the process and is proportional to the level of the interest rate. It decreases as the interest rate $r_t \to 0$. The second asset is a risky security (the stock) . The pension fund takes these prices as given and chooses quantities without transaction costs. Further assumptions are that the securities pay no dividend and taxes on capital gains are zero. Throughout this paper, we are assume a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a filtration $\{\mathcal{F}_t\}$.Uncertainty in the models is generated by standard Brownian motion Z_t. The two equations governing the dynamics of the money market account (bond) and stock are now given as;

$$dB_t = (\alpha + \sigma t)B_t dt$$

or

$$B_t = B_o exp\left\{\alpha t + \frac{\sigma t^2}{2}\right\} \tag{1}$$

and

$$dS_t = \mu S_t dt + \sigma S_t dZ_t$$

or

$$S_t = S_o exp\left\{\sigma Z_t + \left(\mu - \frac{\sigma^2}{2}\right) t,\right\}, \forall \; t \in [0, 1] \tag{2}$$

The admissible trading strategies are (D, I) . The processes D and I are cumulative amount of sales and purchases of stock. The two processes satisfy $D(0) = I(0) = 0$, and both are non-decreasing, right continuous adapted. The evolution of the amount invested in the money market account and stock can be expressed as (Osu and Ihedioha, 2011):

$$\left.\begin{array}{l} dB_t = (\alpha + \sigma t)B_t dt - dI_t + dD_t \\ dS_t = \mu S_t dt + \sigma S_t dZ_t + dI_t - dD_t \end{array}\right\}. \tag{3}$$

Pension funds all face a risky return trade-off of providing a safe pension at low cost. The decision making in a multiple member and multiple objective pension plan depends on the pension funds government, the financial position of the fund and risk attitudes and solvency positions, indexation quality and assets-liability risks are considered as primary objectives, Miao (2010) . For tractability, quantitative derivation and insightful analytic solutions to optimal investment of a pension fund, we use CRRA UTILITY function of the final wealth, that is, $U(W) = \frac{W^{1-\gamma}}{1-\gamma}, for\ 0 < \gamma < 1. \gamma$ is the constant relative risk aversion parameter (that is the relative risk premium) .

On behalf of the plan participants, the pension fund chooses optimal investment strategies D and I so as to maximize the final wealth at a deterministic time T.

Define the value function at time T as;

$$V(B, S, t, T) = \text{Max}_{(D,I)}\ E\left[\frac{(B_T + S_T)^{1-\gamma}}{1-\gamma}\right],$$

where $W = B_T + S_T$ is the total investment from both the riskless and the risky assets.

Assumption 1

The parameter values satisfy:

$$0 < \frac{\mu - (\alpha + \sigma t)}{\gamma \sigma^2} < 1.$$

It guarantees that B and S would be chosen to be strictly positive.

The pension funds problems can therefore be written as:

$$V(B, S, t, T) = \text{Max}_{(S_t; t>0)}\ E\left[\frac{(B_T + S_T)^{1-\gamma}}{1-\gamma}\right] \tag{4}$$

Subject to:

$$d(B_t + S_t) = (\alpha + \sigma t)B_t dt + \mu S_t dt + \sigma S_t dZ_t\ . \tag{5}$$

The wealth of the pension fund $W \equiv B + S$ with the dynamics is given by:

$$dW_t = (\alpha + \sigma t)B_t dt + \mu S_t dt + \sigma S_t dZ_t\ . \tag{6}$$

By Itos lemma (Miao, 2010), we have theH-J-B equation;

$$V_t + [(\alpha + \sigma t)B_t dt + \mu S_t]V_W + \frac{\sigma^2 S_t{}^2}{2} V_{WW} = 0, \tag{7}$$

with terminal condition,

$$V(B, S, T, T) = \frac{W_T{}^{1-\gamma}}{1-\gamma}\ . \tag{8}$$

Using the condition, $W = B + S$, we have

$$V_t + \{[(\alpha + \sigma t)W_t] + [\mu - (\alpha + \sigma t)]S_t\}V_W + \frac{\sigma^2 S_t{}^2}{2} V_{WW} = 0. \tag{9}$$

Observing the homogeneity of the objective function, the restriction and the terminal condition, we conjecture that the value function V must be linear to $\frac{W^{1-\gamma}}{1-\gamma}$. Rewrite $V\ (B, S, t, T)$ as $H(t; T)\frac{W^{1-\gamma}}{1-\gamma}$ to replace the value functions in (7) and (9) becomes;

$$V_t = \frac{W^{1-\gamma}}{1-\gamma} H' + W^{-\gamma} H\big[(\alpha + \sigma t)W + [\mu - (\alpha + \sigma t)S]\big]$$
$$-\frac{\gamma}{2} H W^{-1-\gamma} \sigma^2 S^2 = 0 \,. \qquad (10)$$

It is easy to see from (10) that the optimal values of S and B are;

$$S_t = \frac{[\mu-(\alpha+\sigma t)]W}{\gamma\sigma^2} \text{ and } B_t = \frac{[\gamma\sigma^2 + (\alpha+\sigma t) - \mu]W}{\gamma\sigma^2} \,. \qquad (11)$$

Clearly, the optimal investment policy is characterized by

$$h^*\,(t) \;=\; \frac{B^*}{S^*} = \frac{\gamma\sigma^2+(\alpha+\sigma t)-\mu}{\mu-(\alpha+\sigma t)} \,. \qquad (12)$$

Lemma 1

Given the HJB in (7), we have

$$\mathrm{H} = \mathrm{e}^{-\,(1-\gamma)\,\int_t^T\,(\alpha+\sigma\tau)\left[1+\frac{[\mu-(\alpha+\sigma t)]^2}{2\gamma\sigma^2\,(\alpha+\sigma\tau)}\right]d\tau} \,. \qquad (13)$$

Proof

The solution of the HJB in (7) is;

$$\frac{H'}{1-\gamma} + \left\{[(\alpha + \sigma t)] + \frac{[\mu-(\alpha+\sigma t)]^2}{2\gamma\sigma^2}\right\} H = 0, \qquad (14)$$

to which we obtain (13) and together with the terminal condition we have

$$H(T,T) = e^{(\gamma-1)\int_T^T (\alpha+\sigma\tau)\left[1+\frac{[\mu-(\alpha+\sigma t)]^2}{2\gamma\sigma^2\,(\alpha+\sigma\tau)}\right]d\tau}$$
$$= 1. \qquad (15)$$

This implies that the horizon dependent solution to the investment problem is:

$$V(B,S,t,T) = \frac{W^{1-\gamma}}{1-\gamma} e^{-\,(1-\gamma)\,\int_t^T\,(\alpha+\sigma\tau)\,[1+]\frac{[\mu-(\alpha+\sigma t)]^2}{2\gamma\sigma^2\,(\alpha+\sigma\tau)}d\tau} \,. \qquad (16)$$

This is the maximized lifetime expected utility at time t under optimal investment policy, and at terminal date T, $V(B,S,T,T) = \frac{W^{1-\gamma}}{1-\gamma}$ as expected.

Case of Proportional Transaction Costs

Constantinides (1979, 1986) and Taksar et al (1983), among others had shown that an investment policy is simple in a sense that it is characterized by two reflecting barriers, the buy boundary P and the sell boundary O with $O < P$.The investor stops transacting as far as the portfolio ratio $\frac{B_t}{S_t}$ falls in the no-transaction region$[O, P]$, while it immediately transacts to the closest boundary when the ratio falls outside. In line with the proportional nature of transaction costs, the optimal trading size in continuous time model is always infinitely small so as to keep the portfolio ratio in the interval of no-transaction region.

To capture the idea that purchasing stock and bond both involves transaction costs, the proportional transaction cost rate q, is defined as the amount of one asset the investor can buy by selling one unit amount of the other. This definition reflects the two-way property of transaction costs. Restating the pension fund's problem, the finite horizon value function becomes:

$$V\ (B,S,t;T) = \text{Max}_{(B_t,S_t:t>0)}\ E\left[\frac{(B_T+S_T)^{1-\gamma}}{1-\gamma}\right] \tag{17}$$

subject to:

$$dB_t = (\alpha+\sigma t)B_t dt - dI_t + (1-k)dD_t$$
$$dS_t = \mu S_t dt + \sigma S_t dZ_t + (1-k)dI_t - dD_t$$

Assumption 2

The value function $V\ (B,S,t;\ T)$ is once continuously differentiable in B and twice continuously differentiable in S.

At each point in time, the three regions are identified depending on the portfolio ratio: the no-transaction region, the buy region, and the sell region (Davis and Norman, 1990; Liu and Loewenstein, 2002) . At the no-transaction region, the value function via Ito's formula must satisfy the modified HJB equation:

$$V_t + V_B\ (\alpha+\beta t)\ B + V_S \mu s + \frac{1}{2} V_{SS}\sigma^2 s^2 = 0,\ O \leq \frac{B}{S} \leq P. \tag{18}$$

For $\frac{B}{S} > P$, in the buy region, the marginal rate of substitution of the money market account for the stock must equal $\frac{1}{1-q}$:that is, when in the buy region, sell dI amount of B to purchase $(1-q)\ dI$ amount of S. The value matching condition is satisfied such that there is no jump in the value problem.

$$V_B = (1-q)V_S, \frac{B}{S} > P\ . \tag{19}$$

For the $B/S < A$, in the sell region, the marginal substitution of the money market account for the stock must equal $(1-q)$, such that

$$(1-q)V_B = V_S, \frac{B}{S} < O. \tag{20}$$

In addition, the terminal condition must be satisfied. The terminal condition reflects the fact that at the time the representative individual retires (the horizon point), the pension fund has to turn all the investment into cash and pay out as pension benefits. It is difficult to solve this system of partial differential equations (PDEs) . We introduce a new value function in order to get an equivalent system of ordinary differential equations (ODEs) . The value function $V\ (B,S,t;\ T)$ is homogeneous of degree $1-\gamma$ for all positive numbers in (B,S), as shown in Fleming and Soner (1993) . Define, $h = \frac{B}{S}$, for a new value function

g: $(-\infty,+\infty) \times [0,T] \to \mathcal{R}$, homogeneity gives:

$$V\ (B,S,t;\ T) = s^{1-\gamma} g\ (h,t;T)\ . \tag{21}$$

The no-transaction region, buy region and sell region thus can be characterized by two horizon-dependent boundaries $P\ (t;\ T)$ and $O\ (t;\ T)$. By applying the chain rule, we find the new value function and its derivatives with respect to h and t:

On the no-transaction region

$$\frac{1}{2}g_{hh}\sigma^2h^2 + g_h\left(\gamma\sigma^2 - \left(\mu - (\alpha + \sigma t)\right)\right)h + g(1-\gamma)\left(\mu - \frac{\gamma\sigma^2}{2}\right) + g_t = 0,$$
$$O\ (t;T)\ \leq h \leq P\ (t,T)\ . \qquad (22)$$

On the buy region

$$\left(\frac{1}{1-q} + h\right)g_h(h,t{:}T) = (1-\gamma)g(h,t.T), h > P\ (t,T)\ . \qquad (23)$$

On the sell region

$$(1-q+h)g_h(h,t{:}T) = (1-\gamma)g(h,t;T), h < O\ (t,T), \qquad (24)$$

and the terminal condition to be satisfied is:

$$g(h,T;T) = \frac{(h_T+1)^{1-\gamma}}{1-\gamma}. \qquad (25)$$

In the infinite horizon models, the search for the limit is identical to the search for the stationary solution, for which: (i) the value function $V\ (B,S,t;T)$ is independent of time t; (ii) the buy boundary P and the sell boundary O are independent of time, in addition to being independent of B and S (Dumas and Luciano, 1991) .

In contrast, the finite horizon model highlights horizon dependence of the value function $g\ (h,t;\ T)$ and the corresponding two boundaries $O(t;\ T)$and $P\ (t;\ T)$. Since the two free boundaries are moving over time. It is difficult to solve the PDEs system depicted above but a closed form solution is offered in what follows.

Theorem 1: Let $g\ (h)$ be the value function of the pension funds with h as the prevailing money market account-stock ratio. Let $g\ (h)$ be twice continuously differentiable, the solution of the time-homogeneous value function equation (22) with ;

$$h(0) = 0, \text{and } g'(h) = 0 \qquad (26)$$

is given by;

$$g\ (h)\ = c\ (\sigma h)^{\ \lambda_1} + \frac{h}{k - \frac{\xi(t)}{\sigma}}, \qquad (27)$$

with

$$\frac{\hat{h}}{k - \frac{\xi(t)}{\sigma}} + c\lambda_1\left(\sigma\hat{h}\right)^{\lambda_1} = 0 \qquad (28)$$

and

$$g(h) = \frac{1}{\xi(t) - \sigma k}\{\frac{\sigma^{1+\lambda_1}\dot{\mathrm{h}}^{1-\lambda_1}h^{\lambda_1}}{\lambda_1} + \sigma h\}, \qquad (29)$$

where $\hat{h}$is the expected optimal money market account-stock ratio for a period t, c is a constant and;

$$\lambda_1 = -\left[\frac{\xi(t)}{\sigma} - \frac{1}{2}\right] + \left\{\left[\frac{\xi(t)}{\sigma} - \frac{1}{2}\right]^2 + 2k\right\}^{\frac{1}{2}} \tag{30a}$$

$$\lambda_2 = -\left[\frac{\xi(t)}{\sigma} - \frac{1}{2}\right] - \left\{\left[\frac{\xi(t)}{\sigma} - \frac{1}{2}\right]^2 + 2k\right\}^{\frac{1}{2}}, \tag{30b}$$

are the positive and negative characteristic roots of (29) respectively.

Proof

Let $\xi(t) = \left(\gamma\sigma^2 - (\mu - (\alpha + \sigma t))\right)$ and$k = (\gamma - 1)\left(\mu - \frac{\gamma\sigma^2}{2}\right)$, then (22) reduces to the ode (with the conditions in Osu and Okoroafor, 2007)

$$\frac{\sigma^2 h^2}{2} g_{hh} + \xi(t) h g_h - kg = -h \tag{31}$$

By the method of change of independent variables using Euler's substitution and solving by variation of parameters, the solutions obtained (Osu and okoroafor, 2007) . An important relationship derived under the optimal condition is that the discount rate is proxy of the systematic volatility factor in the economy. So that the discounted rate gains from a unit investment at $\hat{h}$ equals the optimal unit $\bar{h}$ of ratio of money market account to stock for the expected optimal money market account to stock ratio $\hat{h}$.Therefore, by (29), we have;

$$g\ (\hat{h})\ = c\ (\sigma\hat{h})^{\ \lambda_1} + \frac{\hat{h}}{k - \frac{\xi(t)}{\sigma}} = \bar{h}\ . \tag{32}$$

Solving for c in (29) and (32) and equating the results gives;

$$\hat{h} = \frac{\lambda_1[k - \frac{\xi(t)}{\sigma}]}{\lambda_1 - 1}\ . \tag{33}$$

Note: When the drift parameter ξ(t) is large enough so that $\frac{\xi(t)}{\sigma} > \frac{1}{2}$ then the right hand side of (30a) is approximated by first order Taylor's expansion as $\frac{k}{\frac{\xi(t)}{\sigma} - \frac{1}{2}}$, thus, $\frac{\xi(t)}{\sigma} - \frac{1}{2} = k$ and we obtain$\lambda_1 = 1$ and the optimal money market account-stock ratio of (42) is indeterminate.

Observe that for $\xi(t) = \sigma k$, (29) becomes;

$$g(h) = \frac{1}{\xi(t)\ (1-\sigma)}\left\{\frac{\sigma^{1+\lambda_1}\dot{h}^{1-\lambda_1} h^{\lambda_1}}{\lambda_1} + \sigma h\right\}, \tag{34}$$

in which

$$\sigma_{1,2}(t) = \frac{1}{2}\left\{\frac{-2t}{\gamma(\gamma+1)} \pm \sqrt{\left(\frac{2t}{\gamma(\gamma+1)}\right)^2 + 8\left(\frac{\alpha - \gamma\mu}{\gamma(\gamma+1)}\right)}\right\}, \tag{35}$$

with the

$$\sigma_1(t) = \left\{\frac{-t}{\gamma(\gamma+1)} \pm \sqrt{\left(\frac{t}{\gamma(\gamma+1)}\right)^2 + 2\left(\frac{\alpha-\gamma\mu}{\gamma(\gamma+1)}\right)}\right\}. \tag{36}$$

(36) becomes zero for $\alpha = \gamma\mu$ and $\xi(t) = \alpha - \mu$ (the excess return of risky stock over the return from risk-less bond) . Also $g(h) = g(0) = 0$

This implies, if there is no risk, there is no investment and vice versa. Another implication in that there is a zero investment in the money market account and the initial investment in the money market account equals the rate of return of the risky asset (stock) . Again $\sigma_1(t) = 1$if $\gamma = 1, t = T = 0$ and $\alpha = 1 - \mu$ (that is if the mean return of the risky and risk-less investment is unity) . In this case $g(h) = g(\infty) = \infty$.

The investment in the money market account increases by two units while the investment in the risky asset (stock) reduces by one unit. Also a decrement of the investment in the money market account by two units results to an increment in the stock investment by one unit.

On the sell and buy regions we have'

$$\frac{1}{1-q} + h = 1 - q + h, \tag{37}$$

which gives $q = 0$ or $q = 2$.

On the Buy Region

$$g' = \frac{1-\gamma}{\frac{1}{1-q}+h} g\ . \tag{38}$$

The variation of the value function with respect to the ratio of the investments the money market account-stock ratio, h, is $\frac{1-\gamma}{\frac{1}{1-q}+h} g$, for which we have

$$g(h) = k\left(\frac{1}{1-q} + h\right)^{1-\gamma}. \tag{39}$$

Equation (39) becomes $g(h) = k(1+h)^{1-\gamma}$ for $q = 0$ and $g(h) = k(h-1)^{1-\gamma}$, for $q = 2$. For $k > 0$, $g\ (h)$ increases as $q \to 0$ and decreases as $q \to 2$. Thus if there is no transaction cost growth rate of the value function is higher than when transaction cost.

On the Sell Region;

$$g' = \frac{(1-\gamma)}{1-q+h} g. \tag{40}$$

The variation of the value function with respect to the investment ratio h is $\frac{\gamma-1}{1-q+h} g$, which gives

$$g(h) = k(1 - q + h)^{1-\gamma}\ . \tag{41}$$

Generally, the value of the investment increases (decrease) in the sell region and decreases (increases) in the buy region for some values of $0 < q < 2$, but are equal when $q = 0$ or $q = 2$.

Conclusion

In this study the optional investment policy for pension funds when regarding transaction costs is investigated. We have shown that in contrast to the infinite horizon case, the optional investment policy represented by sell and bay boundaries is strongly horizon–dependent for an investor who attempts to maximize his utility on a finite horizon.

The case of no-transaction cost and proportional transaction costs are examined. According to the set of ordinary differential equations derived, for the no-transaction, region, buy and sell regions, it is confirmed that the optimal investment policy is horizon-dependent and closed form solution proffered.

In the no-transaction region, from (38), the value of the investment is zero which implies no risk no investment. Further, there is a zero investment in the money market account leading to equality of the initial investment in the money market account and the role of return of the risky asset (stock) .

A unit increment in variation on the buy region leads to a unit decrement variation on the sell region, vice versa.

It is found that the parameters $\sigma,$ is horizon–dependent so pension funds should take into consideration the horizon dependency when making policy for the participants. It can also be established that α, γ and μ are all horizon dependent.

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