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The Stress: Intensity Factors for Three Griffith-cracks opened by Body Forces in Stress-free Isotropic Rectangle

Dharam Veer Singh[1], Harendra Singh[2], P.S. Kushwaha[3] and Shelendra Kumar[4]

[1]*Department of Mathematics, Research Scholar,*
St. John's College, Agra-280002, Uttar Pradesh, India.
E-mail: ds99yadav@gmail.com
[2,3]*Department of Mathematics, Hindustan College of Science & Technology,*
Farah-281122, Mathura, Uttar Pradesh, India.
[4]*Department of Mathematics, Sanjay Institute of Engineering & Management,*
Chaumunha, Mathura-281406, Uttar Pradesh, India.
E-mail: skumar_maths@rediffmail.com

Abstract

The closed form expressions of stress-intensity factors and of crack shape have been obtained by using finite Fourier sine and cosine transform along with cross-linear superposition principle. A special case of point body Force has been considered.

Keywords: Stress-intensity factors, Crack opening displacement, Fourier Transform.

2000 AMS Subject Classification Number: 74B10, 74R20

Introduction

In the present paper we shall be discussing the analysis due to presence of three interior Griffith-cracks. The cracks occupy the region *y=0*, $0 \le |x| < b, c < |x| < d$ in a rectangle of dimensions 2*a* and 2δ along *x* and *y* axis, respectively. The physical problem is reduced to the following boundary conditions,

$$\sigma_{xx}(a,y) = \sigma_{xy}(a,y) = 0, 0 \le y \le \delta,$$
$$\sigma_{yy}(x,\delta) = \sigma_{xy}(x,\delta) = \sigma_{xy}(x,0) = 0, 0 \le x \le a \qquad (1.1) - (1.5)$$

Where symmetry of geometry has been used. The solution domain will be $[0,a]\cup[0,\delta]$. The mixed-boundary conditions are given below.

$$u_y(x,0)=0, b\le x\le c, d\le x\le a,$$
$$\sigma_{yy}(x,0)=0, 0\le x\le b, c\le x\le d \qquad (1.6) - (1.7)$$

Where middle crack is of length 2*b* and two outer similar cracks are of lengths (d - c).

We shall divide the physical quantities, stress and displacement component at a point (*x, y*) as

$$\sigma_{ij}(x,y)=\sigma_{ij}^{(e)}(x,y)+\sigma_{ij}^{(b)}(x,y), u_i(x,y)=u_i^{(e)}(x,y)+u_i^{(b)}(x,y), \; ij=x,y$$

The super scripts (b) and (e) correspond to body force and elasticity problem, respectively. The shear stress developed by body forces at (*a, y*), (x,δ), (*x,0*) are zero and $u_y^{(b)}(x,0)=0$

Thus the boundary conditions (1.1) – (1.5) are reduced to,

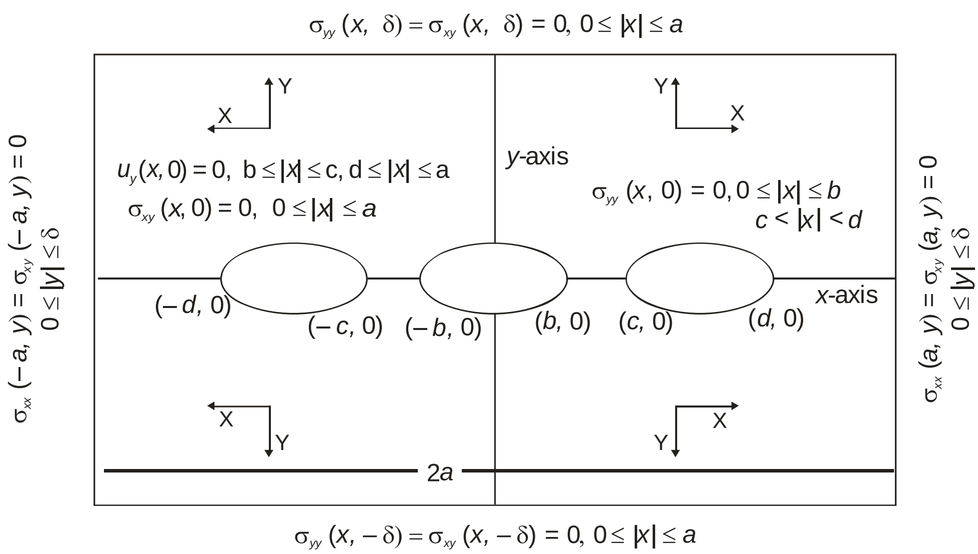
$$\sigma_{xx}^{(e)}(a,y)=-\sigma_{xx}^{(b)}(a,y), \sigma_{xy}^{(e)}(a,y)=0, \; 0\le y\le \delta \qquad (1.8) – (1.9)$$
$$\sigma_{yy}^{(e)}(x,\delta)=-\sigma_{yy}^{(b)}(x,\delta), \sigma_{xy}^{(e)}(x,\delta)=0, \sigma_{xy}^{(e)}(x,0)=0, \; 0\le x\le a \qquad (1.10) – (1.12)$$

And mixed-boundary conditions,

$$u_y^{(e)}(x,0)=0, \; b\le x\le c, \; d\le x\le a; \sigma_{yy}^{(e)}(x,0)=-\sigma_{yy}^{(b)}(x,0), 0 < x < b, d < x < e, \qquad (1.13)-(1.14)$$

It is observed throughout; see Burniston [1],

$$u_y^{(e)}(x,0)=0, \; 0\le |x|\le b, \; c<|x|<d, \qquad (1.15)$$



(FIGURE- 1. Geometry of Problem with Boundary Conditions.)

Microcrack interaction with a main crack obtained by Rose [8]. Tanigawa and Subra [9] developed Thermal stress analysis of a rectangular plate and its thermal stress intensity factor for compressive stress field. Ken et al [3] analysed Stress intensity factor doe to an edge crack in an anisotropic elastic solid. Evaluation of crack tip fields and stress intensity factors in functionally graded elastic materials: cracks parallel to elastic gradient investigated by Rousseau and Tippur [7]. Wang [10] studied Fracture mechanics analysis models for functionally graded Materials with arbitrarily distributed properties (modes II and III problems). Thermal stress intensity factors for a normal crack in multilayered medium has found by Kadaoayler [5]. Recently, Rousseau et al [2] discussed Experimental Fracture Mechanics of Functionally Graded Materials: An Overview of Optical Investigations.

In this present study, the rectangle is assumed in plane strain condition. The body forces are symmetrical. The plan of the paper is as follows: In section-2 we will give solution of body force problem and will reduce the problem to quadruple series equation by solving elasticity problem see Sneddon [4]. The solution of above series equation will be given in section 3 see Kushwaha [6]. The physical quantities of interest will be given in section 4. The solution of Fredholm integral equation for point body force will be given section 5. Discussion and Conclusion will be in section 6.

Solution of Problem

Body force Problem

The problem of body force is obtained by taking the finite Fourier sine & Cosine transforms of equations of equilibrium in the presence of body forces (X,Y). Then using the stress-strain relations, after taking Fourier transform of these, in the transformed relations of equations of equilibrium. We will get two algebraic equations in transform of u_x and u_y. Then solving these equations for u_{xcs} and u_{ysc}. Thus, after Fourier inversion, we get stress-components $\sigma_{xx}^{(b)}$, $\sigma_{yy}^{(b)}$, $\sigma_{xy}^{(b)}$ and displacement components $\left(u_x^{(b)}, u_y^{(b)}\right)$-

$$u_x^{(b)}(x,y) = \frac{1}{\delta} u_{xc}^{(b)}(x,0) + \frac{2}{\delta}\sum_{m=1}^{\infty} u_{xc}^{(b)}(x,\beta_m)\cos(\beta_m y) \tag{2.1}$$

$$u_y^{(b)}(x,y) = \frac{1}{\alpha} u_{yc}^{(b)}(0,y) + \frac{2}{\alpha}\sum_{n=1}^{\infty} u_{yc}^{(b)}(\alpha_n,y)\cos(\alpha_n x) \tag{2.2}$$

$$\sigma_{xx}^{(b)}(x,y) = \frac{4}{\delta\alpha}\left[\frac{\sigma_{xxcc}^{(b)}}{4}(0,0) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sigma_{xxcc}^{(b)}(\alpha_n,\beta_m)\cos(\alpha_n x)\cos(\beta_m y)\right] \tag{2.3}$$

$$\sigma_{yy}^{(b)}(x,y) = \frac{4}{\delta\alpha}\left[\frac{\sigma_{yycc}^{(b)}(0,0)}{4} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sigma_{yycc}^{(b)}(\alpha_n,\beta_m)\cos(\alpha_n x)\cos(\beta_m y)\right] \tag{2.4}$$

$$\sigma_{xy}^{(b)}(x,y) = \frac{4}{\delta\alpha}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sigma_{xyss}^{(b)}(\alpha_n,\beta_m)\sin(\alpha_n x)\sin(\beta_m y) \tag{2.5}$$

$$u_{xc}^{(b)}(x,\beta_m) = \frac{2}{\alpha}\sum_{n=1}^{\infty} u_{xsc}^{(b)}(\alpha_n,\beta_m)\sin(\alpha_n x),\ u_{yc}^{(b)}(\alpha_n,y)\frac{2}{\delta}\sum_{n=1}^{\infty} u_{ycs}^{(b)}(\alpha_n,\beta_m)\sin(\beta_n y) \tag{2.6a}$$

$$\sigma_{xxcc}^{(b)} = \frac{\rho}{w_5}\left[X_{sc}w_6 + Y_{cs}w_7\right],\ \sigma_{yycc}^{(b)} = \frac{\rho}{w_5}\left[X_{sc}w_8 + Y_{cs}w_9\right],\ \sigma_{xyss}^{(b)} = -\frac{\rho}{w_5}\left[X_{sc}w_{10} + Y_{cs}w11\right] \quad (2.6b)$$

$$\left.\begin{aligned} & w_6 = \mu\alpha_n\left[\left(\lambda+2\mu\right)\alpha_n^2 + (3\lambda+\mu-\mu)\beta_m^2\right],\ w_7 = \mu\beta_m\left[\left(\lambda+2\mu\right)\alpha_n^2 - \lambda\beta_m^2\right], \\ & w_8 = \alpha_n\mu\left[\lambda\alpha_n^2 - \left(\lambda+2\mu\right)\beta_m^2\right],\ w_9 = \mu\beta\left[(\lambda+2\mu)\beta_m^2 + \left(3\lambda+4\mu\right)\alpha_n^2\right], \\ & w_{10} = -\mu\beta_m\left\langle(\lambda+2\mu)\beta_m^2 - \lambda\alpha_n^2\right\rangle,\ w_{11} = -\mu\alpha\left\langle(\lambda+2\mu)\alpha_n^2 - \lambda\beta_m^2\right\rangle \end{aligned}\right\} \quad (2.6c)$$

Solution of Elasticity Problem

The solution of elasticity problem is obtained by the method of Airy's stress function method see Sneddon [4], and then using cross-linear-superposition principle. We get-

$$u_x^{(e)}(x,y) = \frac{1}{2}u_{xc}^{(e)}(x,0) + \sum_{m=1}^{\infty}\cos(\beta_m y)u_{xc}^{(e)}(x,\beta) + \sum_{n=1}^{\infty}\sin(\alpha_n, x)u_{xs}^{(e)}(\alpha_n, y), \quad (2.7)$$

$$u_y^{(e)}(x,y) = \frac{1}{2}u_{yc}^{(e)}(0,y) + \sum_{n=1}^{\infty}\cos(\alpha_n, x)u_{yc}^{(e)}(\alpha_n, y) + \sum_{m=1}^{\infty}\sin(\beta_m y)u_{ys}^{(e)}(x,\beta_m) \quad (2.8)$$

Where -

$$u_{xc}^{(e)}(x,\beta_m) = \left[\frac{(1-\eta)G_{xxx}(x,\beta_m) - \eta\beta_m^2 G_x,(x,\beta_m)}{\beta_m^2}\right],$$

$$u_{xs}^{(e)}(\alpha_n, y) = \left[\frac{(1-\eta)H_{yy}(\alpha_n, y) - \eta\alpha_n^2 H}{\alpha_n}\right] \quad (2.9)$$

$$u_{yc}^{(e)}(\alpha_n, y) = \left[\frac{(1-\eta)H_{yyy},(\alpha_n, y) - \eta\alpha_n^2\alpha_n, y}{\alpha_n^2}\right], u_{ys}^{(e)}(x,\beta_m)$$

$$= \left[\frac{(1-\eta)G_{xxx},(x,\beta_m) - \eta\beta_m^2 G(x,\beta_m)}{\beta_m}\right] \quad (2.10)$$

$$H(\alpha_n, y) = (A + yB)\cosh(\alpha_n y) + (C + yD)\sinh(\alpha_n, y),$$

$$G(x,\beta_m) = (E + xF)\cosh(\beta_m x) \quad (2.11)$$

Where A, B, C, D, E, & F are six unknown constants which are to be determined by six boundary conditions.

$$\sigma_{xy}^{(e)}(x,y) = \sum_{n=1}^{\infty}\alpha_n\sin(\alpha_n x)\begin{bmatrix}\alpha_n(A + yB)\sinh(\alpha_n y) \\ +B\cosh(\alpha_n y) + \alpha_n(C + yD) \\ \cosh(\alpha_n y) + D\sinh(\alpha_n y)\end{bmatrix} + \sum_{m=1}^{\infty}\beta_m\left[\beta_m(E + xF)\sinh(\beta_m x)\right] \quad (2.12)$$

$$\left.\begin{aligned} & \sigma_{yy}^{(e)}(x,y) = -\sum_{n=1}^{\infty}\alpha_n^2\left[(A + yB)\cosh(\alpha_n y) + (C + yD)\sinh(\alpha_n y)\right]\cos\alpha_n x \\ & +\sum_{m=1}^{\infty}\left[\beta_m^2(E + xF)\cosh(\beta_m x) + 2F\beta_m\sinh\beta_m x\right]\cos\beta_m y \end{aligned}\right\} \quad (2.13)$$

$$\left.\begin{aligned} & \sigma_{xx}^{(e)}(x,y) = \sum_{n=1}^{\infty}\left[\alpha_n(A + yB)\cosh(\alpha_n y) + 2B\sinh(\alpha_n y) + \alpha_n(C + yD)\sinh(\alpha_n y) + 2D\cosh(\alpha_n y)\right] \\ & \cos(\alpha_n x) - \sum_{m=1}^{\infty}\beta_m^2(E + xF)\cosh(\beta_m x)\cos(\beta_m y) \end{aligned}\right\} \quad (2.14)$$

The boundary conditions (1.8) – (1.12) will determine five constants in terms of sixth i.e. B.

Reduction to Quadruple Series Equation

The mixed condition (1.13-1.14) and (1.15) give:

$$2(1-\eta)\left[\frac{u_1}{2}+\sum_{n=1}^{\infty}B\cos\alpha_n x\right]=0,\ b\le x\le c, d\le x\le a\ ,\ u_1=\frac{u_0}{2(1-\eta)}=\text{ constant} \tag{2.15}$$

$$\sum_{n=1}^{\infty}\alpha_n B\cos(\alpha_n x)=-\sigma_{yy}^{(b)}(x,0)+p_{11}(x)-\sum_{n=1}^{\infty}p_{12}(n,x)B,\ 0\le x<b, c<x<d \tag{2.16}$$

$$p_{12}(n,x)=p_8(n,x)+p_{10}(n,x)+(-1)^n p_9(n,x) \tag{2.16a}$$

$$\left.\begin{aligned}
&p_{11}(x)=p_5(x)+p_6(x)-p_7(x), p_{10}(n,x)=\alpha_n\frac{(a_{15}-a_{16})}{a_{16}}\cos(\alpha_n x)\\
&p_9(n,x)=\sum_{m=1}^{\infty}\frac{\beta_m t_2}{t_m a}\left[\beta_m(1+x\alpha_{11})\cosh(\beta_m x)+2a_{11}\sinh(\beta_m x)\right]\\
&p_8(n,x)=\cos\alpha_n x\sum_{m=1}^{\infty}\frac{(-1)^m a_{14}(m)}{t_m a}\beta_m^2\cosh\beta_m a\frac{t_2(n,m)}{\alpha_n^2+\beta_m^2}\\
&p_7(x)=\sum_{m=1}^{\infty}\frac{\beta_m t_1(m)}{t_m a}\left[\beta_m(1+xa_{11})\cosh\beta_m(x)+2a_{11}\sinh\beta_m x\right]\\
&p_6(x)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^n}{t_m a}\beta_m^2 a_{14}(m)\cosh\beta_m a t_1(m)\frac{a_1\cos(\alpha_n x)}{a_{16}(\alpha_n^2+\beta_m^2)}, p_5(x)=\sum_{n=1}^{\infty}\frac{\alpha_n a_1}{a_{16}}d_1(\alpha_n)\cos(\alpha_n x)
\end{aligned}\right\} \tag{2.16b}$$

Thus the problem is reduced to the solution of quadruple series relation given by (2.15) & (2.16).

Solution of Quadruple Series Equation

The solution of $\frac{u_1}{2}+\sum_{n=1}^{\infty}B\cos\alpha_n x=0, b\le x\le c, d\le x\le a,$ (3.1)

$$\sum_{n=1}^{\infty}\alpha_n B\cos(\alpha_n x)=-\sigma_{yy}^{(b)}(x,0)+p_{11}(x)-\sum_{n=1}^{\infty}Bp_{12}(n,x)\ ,\ 0\le x<b, c<x<d \tag{3.2}$$

Will be obtained by the method of Kushwaha [3]. We take trial solution as

$$\alpha_n B=2\left[\left\langle\int_0^b g(t)+\int_c^d h(t)\right\rangle\sin(\alpha_n t)dt\right],\ \pi u_1=2\left[\left\langle\int_0^b g(t)+\int_c^d h(t)\right\rangle+dt\right], \qquad (3.3)\text{ - }(3.4)$$

Use of $\frac{qt}{2}+\sum_{n=1}^{\infty}\frac{\sin(\alpha_n t)\cos(\alpha_n x)}{n}=\begin{cases}\frac{a}{2}, t>x\\ 0, t<x\end{cases}$

and (3.1) will be satisfied identically if,

$$\int_e^d h(t)dt=0, \tag{3.5}$$

The substitution of (3.3) into (3.2) and then using the method of Kushwaha [7] we get,

$$g(t) = \frac{2}{a^2}\frac{\sin(qt/2)}{\psi_0(t)}\left[\Delta_0(t) + \left(\int_0^b g(\alpha) - \int_c^d h(\alpha)\right)k(\alpha,t)d\alpha\right], 0 \le t < b, \quad (3.6)$$

$$h(t) = \frac{2}{a^2}\frac{\sin(qt/2)}{\psi_0(t)}\left[\Delta_0(t) + \left(\int_0^b g(\alpha) - \int_c^d h(\alpha)\right)k(\alpha,t)d\alpha\right], c < t < d, \quad (3.7)$$

$$\Delta_0(t) = \left\langle \int_0^b - \int_c^d \right\rangle \frac{\cos(qx/2)}{G(x,t)}\psi_0(x)P_{13}(x)dx + D \quad (3.8)$$

$$K(\alpha,t) = \left\langle \int_0^b - \int_c^d \right\rangle \frac{\cos(qx/2)\psi_0(x)}{G(x,t)}M(x,\alpha)dx \quad (3.9)$$

$$M(x,\alpha) = 2\sum_{n=1}^{\infty} p_{12}(n,x)\sin(\alpha_n x)\cos(\alpha_n x), P_{13}(x) = -\sigma_{yy}^{(b)}(x,0) + p_{11}(x) \quad (3.10)$$

While $p_{11}(x)$, $p_{12}(x)$, are given by (2.16b) and (2.16a), respectively. and,

$$\psi_0(x) = \left|G(x,b)G(x,c)G(x,d)^{1/2}\right|, G(x,b) = \cos(qx) - \cos(qb), q = \pi / a_0 \quad (3.11)$$

D is an arbitrary constant which will be determined though (3.5) and the solution of coupled Fredholm integral equation of second kind given by (3.6) – (3.7).

Physical Quantities

The physical quantities of interest in fracture mechanics are crack opening displacement (COD) and stress-intensity factors (SIF).

Crack Shape

The crack shape is plot of $u_y^{(e)}(x,0)$, against *x*. $u_y^{(e)}(x,0)$, is obtained from the values of series (2.15) for, $0 \le x \le b$, $c \le x \le d$. Thus using (3.3) – (3.4) in (2.15), we get,

$$u_y^{(e)}(x,0) = 2(1-\eta)\begin{cases}\int_x^b g(t)dt, 0 \le x \le b \\ \int_x^d h(t)dt, c \le x \le d\end{cases} \quad (4.1)$$

Where *g(t)* and *h(t)* are solutions of Fredholm integral equation (3.6) – (3.7).

Normal Stress

The normal stress $\sigma_{yy}^{(e)}(x,0)$ is obtained from the value of series (3.2), keeping right hand side on left hand side for $b<x<c$ and $d<x<a$, and then using (3.3) we get,

$$\sigma_{yy}^{(e)}(x,0) = \left\langle \int_0^b g(t) + \int_c^d h(t) \right\rangle \frac{\sin qt}{G(x,t)}dt - P_{11}(x) + F_1(x) \quad (4.2)$$

$$F_1(x) = \left(\int_0^b g(d) - \int_c^d h(\alpha)\right)M(\alpha,x)d\alpha,$$

$$M(\alpha,x) = 2\sum_{n=1}^{\infty} p_{12}\frac{(n,x)\sin(\alpha\alpha_n)\cos(\alpha_n,x)}{\alpha_n} \quad (4.3)-(4.4)$$

Now using the values of *g(t)* and *h(t)* from (3.6) and (3.7) and evaluating the integrals.

$$\sigma_{yy}^{(e)}(x,0) = \left\{ \frac{2}{a} \frac{\sin(qx/2)}{\psi_0(x)} \left[\Delta_0(x) + \left(\int_0^b g(\alpha) - \int_c^d h(\alpha) \right) - P_{11}(x) + F_1(x) \right\}, \; b<x<c \right.$$

$$\sigma_{yy}^{(e)}(x,0) = -\left\{ \frac{2}{a} \frac{\sin(qx/2)}{\psi_0(x)} \left[\Delta_0(x) + \left(\int_0^b g(\alpha) - \int_c^d h(\alpha) \right) K(\alpha,x) d\alpha \right] - P_{11}(x) + F_1(x) \right\},$$

$$d<x<a \qquad (4.5)$$

Stress-Intensity Factor

The stress-intensity factors at crack tips are defined as :

$$K_b = \lim_{x \to b^+} \sqrt{x-b}\,\sigma_{yy}^{(e)}(x,0), \; K_c = \lim_{x \to c^-} \sqrt{c-x}\,\sigma_{yy}^{(e)}(x,0), \; K_d = \lim_{x \to d^+} \sqrt{x-d}\,\sigma_{yy}^{(e)}(x,0) \qquad (4.6)\text{-}(4.8)$$

Using (4.5) in (4.6) – (4.8) and evaluating the limits we get

$$K_b = \psi_2(b)\Delta_2(b), K_c = \psi_2(c)\Delta_2(c), K_d = -\psi_2(d)\Delta_2(d) \qquad (4.9)$$

$$\Delta_2(x) = \Delta_0(x) - \left(\int_0^b g(\alpha) - \int_c^d h(\alpha) \right) K(\alpha,x) d\alpha \qquad (4.10)$$

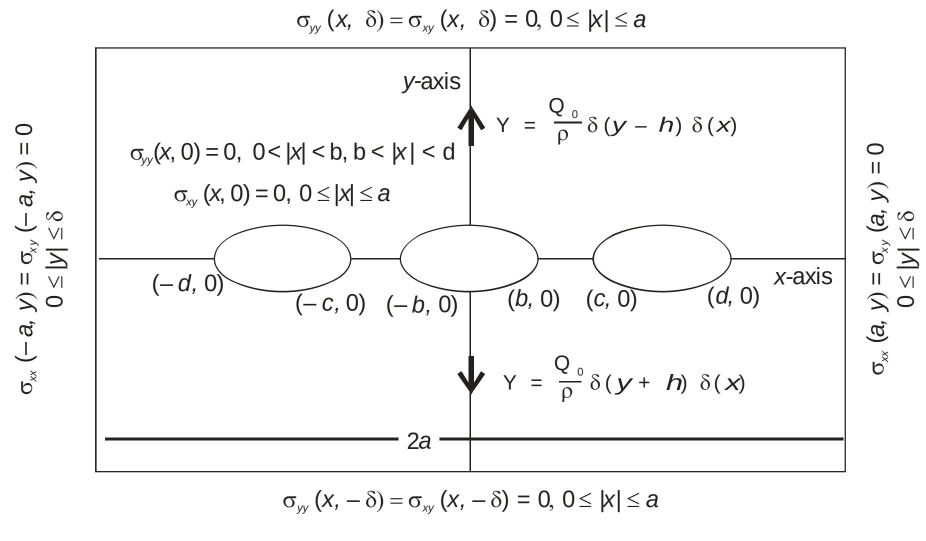
$$\psi_2(x) = 2\sqrt{\frac{\tan(qx/2)}{\pi a}} . \frac{1}{\delta(x)}, \; \delta(x) = \left| \langle \cos(qx) - \cos(qc) \rangle \langle \cos(qx) - \cos(qd) \rangle \right| \qquad (4.11)\text{-}(4.12)$$

Solution of Fredholm Integral Equation

Point Body Force

The rivets are very commonly used. Therefore, it is of practical importance to find out the physical quantities, used in fracture mechanics, due to point body forces. We discuss the point body force, see figure-2

$$Y(x,y) = \frac{Q_0}{2\rho} \left[\delta(y-h) - \delta(y+h) \right] \delta(x), X(x,y) = 0 \qquad (5.1)$$



(FIGURE- 2. Geometry of Problem with Special Point Body Force)

Above point force is of magnitude Q_0 (constant) and acting at $(0,\pm h)$ in positive & *negative y*-directions respectively. And ρ is mass density of the medium.

The finite Fourier cosine and sine transform of Y with respect to x, y, respectively yields,

$$Y_{cs}(\alpha_n,\beta_m)=\frac{Q_0}{\rho}\sin(\beta_m h) \tag{5.2}$$

Now we evaluate $\sigma_{yy}^{(b)}(x,0)$

$$\sigma_{yy}^{(b)}(x,0)=\frac{4Q_0}{a\delta}\left[\psi_2\theta+\frac{1}{2(1-\eta)}\frac{d}{dx}\psi_4(\theta)\right],$$

$$\psi_2\theta=\sum_{i=1}^{4}\psi_1(\theta^{(i)}),\psi_4(\theta)=\sum_{i=1}^{4}\psi_1\theta^{(i)}\psi_{41}(\theta^{(i)}) \tag{5.3}$$

$$\psi_1(\theta^{(i)})=\frac{\sinh\left\{q(\theta)^i\right\}}{\cosh\left\{q(\theta)^i\right\}-\cos(qx)},\psi_{41}(\theta^{(i)})=\frac{\sinh\left\{q(\theta)\right\}}{\cosh\left\{q(\theta)^i\right\}-\cos(qx)} \tag{5.4}$$

$$\theta^{(1)}=h,\theta^{(2)}=h+2\delta,\theta^{(3)}=4\delta-h,\theta^{(4)}=2\delta+h, \tag{5.5}$$

The evaluation of $p_{11}(x)$ and $p_{12}(x)$ is done by expanding the function $\langle p_5,p_6,p_7,p_8,p_9,p_{10}\rangle$ in terms of $e^{-nq\delta}$ and retaining up to $e^{-4q\delta}$ i.e. n=1, 2, 3, 4 only.

Solution of Fredholm Integral Equation (FIE)

The solution of Fredholm integral equation is obtained by the method of approximate expansion of functions involved in equation. These expansions are done in term of $\left\{e^{-mq\delta}\right\}$. We retained up to m = 0, 1, 2, 3, 4 only. Before we come to the solution of FIE, we approximate $M(\alpha,x)$ and $K(\alpha,t)$.

$$M(x,\alpha)=e^{-2q\delta}2\sum_{r=0}^{\infty}\left[\begin{array}{l}\left\langle\Delta_5(\alpha,r;x,2\delta)+e^{-2q\delta}\Delta_5(\alpha,r;x,4\delta)\right\rangle\\ \left\{\frac{\pi(9-x)}{16}-\frac{\cos(qx)}{2}\left\langle 1+\cos(2qx)\sin(2q\alpha)e^{-2q\delta}\right\rangle\right.\\ \left.+\frac{a\delta}{4}\left\langle\frac{\pi}{2}R(\alpha,\delta,x)-q_1\log\left|R_0(\delta,x)\right|\right\rangle-\frac{\pi q}{4}R_1(\alpha,\delta,x)R(\alpha,\delta,x)\sin(qx)\right\}\end{array}\right] \tag{5.6}$$

$$\Delta_5(\alpha,r;x,y)=\frac{d^{2r}}{dy^{2r}}\left\langle R^{\pm}(\alpha,y,x)\right\rangle,\ R^{\pm}(\alpha,y,x)=2\cos(qx)\sin(qx)e^{-qy}R_2(\alpha,y,x) \tag{5.7}$$

$$\left.\begin{array}{l}R_1(\alpha,y,x)=(1-e^{-2qy})R_2(\alpha,y,x),R_0(y,x)=\cosh(qy)-\cos(qx)\\ R_2(\alpha,y,x)=1+2\cos(qx)\sin(qx)e^{-qy}+\cos(2qx)\sin(2q\alpha)e^{-2qy}-4\cos(q\alpha)\sin(q\alpha)e^{-3qy}\end{array}\right\} \tag{5.8}$$

Now we evaluate $K(\alpha,t)$ with, $0\le\alpha,t<b$ for *g* (α) and c< α, *t<d* for *h* (α),

$$K(\alpha,t)=8q^2\sin(q\alpha)\left[T_1(t,\alpha)e^{-2q\delta}-T_2(t,\alpha)e^{-3q\delta}+T_3(t,\alpha)e^{-4q\delta}+T_4(t,\alpha)e^{-6q\delta}\right], \tag{5.9}$$

$$\psi_2(t)=\frac{\alpha\delta\pi}{2}I_1(t)+I_2(t)+S(t)-\frac{\pi\delta}{4}S_2(t) \tag{5.10}$$

$$\left.\begin{array}{l} T_1(t,\alpha)=\sin(q\alpha)\psi_2(t), T_2(t,\alpha)=\dfrac{\pi q}{2}\sin(q\alpha)I_1(t), \\ T_3(t,\alpha)=I_1(t)-\sin(q\alpha)\sin(2q\alpha)\langle 2I_3(t)-I_1(t)\rangle+2\sin(2q\alpha)I_2(t) \\ T_4(t,\alpha)=2\sin(q\alpha)I_1(t)+\sin(2q\alpha)\langle 2I_3(t)-I_1(t)\rangle \end{array}\right\} \quad (5.11)$$

$$\left.\begin{array}{l} I_n(t)=\left(\int_0^b-\int_c^d\right)\dfrac{\sin(qx)G_1(x)\cos(nqx)}{G(x,t)}, \; S_1(t)=\dfrac{\pi}{16}\left(\int_0^b-\int_c^d\right)\dfrac{(qx)\sin(qx)G_1(x)dx}{G(x,t)} \\ S_2(t)=\left(\int_0^b-\int_c^d\right)\dfrac{\sin(qx)G_1(x)\log|R_0(\delta_1 x)|}{G(x,t)}dx \end{array}\right\} \quad (5.11a)$$

Now we assume

$$g(t)=\sum_{m=0}^{\infty}g_m(t)e^{-mq\delta}, 0\le t<b; h(t)=\sum_{m=0}^{\infty}h_m e^{-mq\delta}, c<t<d \quad (5.12)$$

And then substitute (5.12) in (3.6) and (3.7) and then compare the coefficients of $e^{-mq\delta}$ on both sides we get,

$$\left.\begin{array}{l} g_0(t)=\dfrac{2}{a^2}\dfrac{\theta_1(t)\sin(qt/2)}{\psi_0(t)}, 0\le t<b; h_0(t)=\dfrac{2}{a^2}\dfrac{\theta_1(t)\sin(qt/2)}{\psi_0(t)}, c<t<d \\ \theta_1(t)=\left(\int_0^b-\int_c^d\right)\dfrac{\cos(qx/2)\psi_0(x)P_{13}(x)}{G(x,t)}+\dfrac{(2+5\eta)}{8(1+\eta)\delta}I_0(t)+D, \; g_1(t)=h_1(t)=0 \\ P_{13}(x)=\sigma_{yy}^{(b)}(x,0)+P_6(x)-P_7(x) \end{array}\right\} \quad (5.13)$$

$$\left.\begin{array}{l} g_2(t)=\dfrac{2}{a^2}\dfrac{\sin(qt/2)}{\psi_0(t)}\theta_2(t), 0\le t<b; h_2(t)=\dfrac{2}{a^2}\dfrac{\sin(qt/2)}{\psi_0(t)}\theta_2(t), c<t<d \\ \theta_2(t)=\dfrac{2}{\delta}\sum_{i=0}^{2}e_iI_i(t)+R_0(t)+\dfrac{a}{2}R_{01}(t), \; R_0(t)=\sum_{i=0}^{2}\left(\int_0^b g_0(\alpha)-\int_c^d h_0(\alpha)\right)\Delta_5(\alpha,l;y,\delta)d\alpha, \end{array}\right\} \quad (5.14)$$

$$\left.\begin{array}{l} \Delta_5 \text{ is defined is (5.7).}, \; R_{01}(t)=T_2(t)-\dfrac{I_1(t)}{2}, R_2(t)=-\dfrac{\pi}{2}\delta_1(t) \\ g_3(t)=\dfrac{2}{a^2}\dfrac{\sin(qt/2)}{\psi_0(t)}\theta_3(t), 0\le t<b, h_3(t)=\dfrac{2}{a^2}\dfrac{\sin(qt/2)}{\psi_0(t)}\theta_3(t), c<t<d \\ \theta_3(t)=2I_2(t)R_3, \; R_3=\left(\int_0^b g_0(\alpha)-\int_c^d h_0(\alpha)\right)\sin^2(q\alpha)d\alpha, \end{array}\right\} \quad (5.15)$$

$$\left.\begin{array}{l} g_4(t)=\dfrac{2}{a^2}\dfrac{\sin(qt/2)}{\psi_0(t)}\theta_4(t), 0\le t<b, h_4(t)=\dfrac{2}{a^2}\dfrac{\sin(qt/2)}{\psi_0(t)}\theta_4(t), c<t<d \\ \theta_4(t)=T_4(t)+2R_1\{2I_3(t)-I_1(t)\}+2R_2I_1(t), \\ R_1=\left(\int_0^b g_0(\alpha)-\int_c^d h_0(\alpha)\right)\sin(q\alpha)\sin(2q\alpha)d\alpha, \\ R_2=\left(\int_0^b g_2(\alpha)-\int_c^d h_2(\alpha)\right)\sin(q\alpha)d\alpha, T_4(t)=I_0^2(t)(\pi-7\eta q\delta) \end{array}\right\} \quad (5.16)$$

To evaluate D we take $h \approx h_0$ only then using second of (5.13) and (3.5) and evaluating the integrals.

$$D = \frac{\int_c^d \frac{\theta_{11}(t)\sin(qt/2)}{\psi_0(t)}dt}{\int_c^d \frac{\sin(qt/2)}{\psi_0(t)}dt},$$

$$\theta_{11}(t) = \left(\int_0^b - \int_c^d\right)\frac{\cos(qx/2)\psi_0(x)P_{13}(x)}{G(x,t)} + \frac{2+5\eta}{8(1+\eta)\eta}I_0(t) \tag{5.17}$$

$$\left.\begin{array}{l} g(t) = \frac{2}{a^2}\frac{\sin(qt/2)}{\psi_0(t)}\Delta_2(t), 0 \le t < b; h(t) = \frac{2}{a^2}\frac{\sin(qt/2)}{\psi_0(t)}\Delta_2(t), c < t < d \\ \Delta_2(t) = \theta_1(t) + \sum_{i=2}^{4}\theta_1(t)e^{-iq\delta} \end{array}\right\} \tag{5.18}$$

The crack shape will be evaluated through (5.18) and (4.1) after evaluating the integral numerically. The stress-intensity factors are to be evaluated though (5.11a) and (4.9). The integrals involved are to be evaluated numerically.

Discussion and Conclusion

1. The closed form expressions for normal-stress components are obtained.
2. It is observed that normal stress components have Cauchy type singularity at crack tips. The stress-intensity factors too are evaluated.
3. The crack opening displacement is smooth at crack tip.
4. The physical quantities are obtained in terms of solution of coupled Fredholm integral equation. The kernel of Fredholm integral equation is function of boundary conditions and mixed-boundary conditions. Through boundary condition it becomes the function of geometric parameters say δ, a, b, c etc. Through mixed-boundary conditions (more than three parts) we get coupled Fredholm integral equation. Authors are not aware about any other numerical or experimental so that we can compare with.

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