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Asymptotic Equivalence of Exponentially Dichotomic Systems

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Abstract

We study the conditions of asymptotic equivalence in a weakly nonlinear systems of exponentially dichotomic ODEs.

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1. Introduction

Let $\mathcal{R}$ be the set of real numbers. Denote $|\cdot|$ to be the euclidean vector norm on $\mathcal{R}^n$, $\|\cdot\|$ is the norm of $n \times n$ matrix. In this work, we study the conditions of asymptotic equivalence in the system of nonlinear ODEs

$$\frac{dy}{dt} = A(t)y + f(t, y) \tag{1}$$

to the linear system

$$\frac{dx}{dt} = A(t)x \tag{2}$$

Definition 1.1. Systems (1) and (2) are asymptotically equivalent for $t \to \infty$ if there exists a one-to-one correspondence between their solutions $x(t)$ and $y(t)$ such that

$$\lim_{t\to\infty} |x(t) - y(t)| = 0$$

The question of asymptotic equivalence was addressed in the works of various authors. The classic result in this direction is Levinson Theorem [8], which gives the conditions of asymptotic equivalence to the system

$$\frac{dy}{dt} = Ay + B(t)y \tag{3}$$

and

$$\frac{dx}{dt} = Ax \tag{4}$$

It states that if all solutions of system (3) are bounded on semi-axes, and $\int\limits_0^\infty \|B(t)\|\, dt < \infty$, then systems (3) and (4) are asymptotically equivalent. For the time-dependent matrix $A(t)$, the similar results were obtained by Winter [12], in the case when all the solutions of (4) are bounded and the following condition holds; namely

$$\lim_{t\to\infty} \inf \int\limits_0^t \sup\,[B(s)]\, ds > -\infty$$

Later Yakubovich [13] studied the conditions of asymptotic equivalence of system (1) and (2) in the case of constant matrix $A(t) \equiv A$ but without imposing the boundness condition on the solutions of system (2).

After the pioneering works mentioned above, the asymptotic equivalence was studied by various authors for different classes of differential equations, including functional, impulsive and stochastic systems [4,6,7,8, and 10].

We also mention the work Akhmet M.U, Tleubergenova M.A. and Zafer A. [1], where the authors study the conditions of asymptotic equivalence of systems (3)-(4) in the case of nonconstant matrix $A(t)$. However, the imposed conditions are rather difficult to verify. The present work generalizes the results of [13].

We study the conditions of asymptotic equivalence of systems (1) and (2) in the case of exponential dichotomy of system (2).

Let $X(t,s)$ be the matriciant of the system (2), *i.e.* the fundamental matrix, normalized with the condition $X(s,s) = I$, where I is identity matrix and $X(t) = X(t,0)$.

Definition 1.2. System (2) is called exponentially dichotomic on $\mathcal{R}$ if one can find two complementing projectors P_1 and P_2, and also positive constants ν_1, ν_2, N_1, N_2 such that the following inequalities hold:

$$\left\| X(t)P_1X^{-1}(s) \right\| \leq N_1e^{\nu_1(t-s)}, \quad t \geq s \tag{5}$$

and

$$\left\| X(t)P_2X^{-1}(s) \right\| \leq N_2e^{\upsilon_2(s-t)}, \quad s \geq t \tag{6}$$

The conditions of exponential dichotomy are well studied (see, e.g. Ateiwi, [2,3], Daleckiy, Kreyn [5], and Mitropolskiy, Samoilenko, Kulik [9]).

We shall now impose the following conditions:

C_1) The matrix $A(t)$ is defined, continuous and bounded on $\mathcal{R}$, so that $a := \sup\limits_{t\in\mathcal{R}} \|A(t)\| < \infty$.

C_2) The vector-valued function $f(t, y)$ is defined and continuous for $t \geq 0$, $y \in \mathcal{R}^n$, satisfies the condition $|f(t, y_1) - f(t, y_2)| \leq \eta(t)\,|y_1 - y_2|$, for all $t \geq 0, y_1, y_2 \in \mathcal{R}^n$, and some nonnegative function $\eta(t)$ defined for $t \geq 0$.

C_3) $a_1 := \int\limits_0^\infty \eta(t)dt < \infty$.

C_4) $f(t, 0) = 0$, for $t \geq 0$.

2. Auxiliary results

In this section, we present the results used in further analysis.

Lemma 2.1. Under the condition C_1, the matriciant $X(t, s)$ of system (2) satisfies the following inequality for arbitrary $t, s \in \mathcal{R}$:

$$\|X(t, s)\| \leq e^{a|t-s|} \tag{7}$$

The proof of this Lemma follows from the integral representation

$$X(t, s) = I + \int\limits_s^t A(\tau)X(\tau, s)d\tau,$$

and Gronwall-Bellman inequality

Lemma 2.2. Under the conditions $C_1 - C_4$, there exists a positive constant a_2, such that any solution of system (2) satisfies the inequality

$$|y(t)| \leq a_2 \, |y(t_0)| \, e^{a(t-t_0)}, \; t \geq t_0. \tag{8}$$

Proof. Clearly, the existence of any solution $y(t)$ of system (2) for $t \geq t_0$ follows from the conditions of the lemma. Such solution must satisfy the integral identity

$$y(t) = X(t,t_0)y(t_0) + \int_{t_0}^{t} X(t,\tau) f(\tau, y(\tau)) d\tau.$$

Therefore

$$|y(t)| \leq \|X(t,t_0)\| \, |y(t_0)| + \int_{t_0}^{t} \|X(t,\tau)\| \, f(\tau, y(\tau)) d\tau$$

Thus by conditions C_2, C_4 and Lemma 2.1, we obtain

$$\begin{aligned} |y(t)| &\leq \|X(t,t_0)\| \, |y(t_0)| + \int_{t_0}^{t} \|X(t,\tau)\| \, \eta(\tau) \, |y(\tau)| \, d\tau \\ &\leq e^{a(t-t_0)} \, |y(t_0)| + \int_{t_0}^{t} e^{a(t-\tau)} \eta(\tau) \, |y(\tau)| \, d\tau. \end{aligned}$$

Now, multiplying the last inequality by $e^{-a(t-t_0)}$, we get

$$|y(t)| \, e^{-a(t-t_0)} \leq |y(t_0)| + \int_{t_0}^{t} e^{-a(\tau-t_0)} \eta(\tau) \, |y(\tau)| \, d\tau.$$

Applying the Gronwall-Bellman inequality, we have

$$|y(t)| \, e^{-a(t-t_0)} \leq |y(t_0)| \, e^{\int_{t_0}^{t} \eta(\tau) d\tau} \leq |y(t_0)| \, e^{a_1},$$

or

$$|y(t)| \leq e^{a_1} \, |y(t_0)| \, e^{a(t-t_0)}.$$

Note that the statement of Lemma 2.2 follows with $a_2 := e^{a_1}$. ■

Denote $X_1(t,s) = X(t)P_1(s)X^{-1}(s)$, and $X_2(t,s) = X(t)P_2(s)X^{-1}(s)$.

Lemma 2.3. The matrices $X(t,s)$, $X_1(t,s)$ and $X_2(t,s)$ satisfy the following relations:

1) $X(t,s) = X_1(t,s) + X_2(t,s)$.

2) $X_i(t,\tau) = X_i(t,s)X_i(s,\tau)$, for all t, s, and τ.

3) $X_i(t,\tau) = X_i(t,s)X_i(s,\tau)$; $i = 1,2$.

Proof. The first statement follows from the definition of the matrices $X_1(t,s)$ *and* $X_2(t,s)$. The equality $X(t,s) = X(t)X^{-1}(s)$ and mutually complementing property or the vectors P_1, P_2, $P_1 + P_2 = I$.

To prove the second statement, we have

$$\begin{aligned} X_i(t,s)X_i(s,\tau) &= X(t)P_iX^{-1}(s)X(s)P_i(s)X^{-1}(\tau) \\ &= X(t)P_i^2X^{-1}(\tau) = X_i(t,\tau). \end{aligned}$$

Finally, the third statement follows analogously. ■

3. Main result

In this section, we prove the key result on the asymptotic equivalence of systems (1) and (2).

Theorem 3.1. Assume that conditions C_1, C_2, and C_4 are satisfied. Also, suppose that system (2) is exponentially dichotomic on $\mathcal{R}$.Additionally, If

$$a_3 := \int_0^\infty e^{at}\eta(t)dt < \infty \tag{9}$$

then systems (1) and (2) are asymptotically equivalent for $t \to \infty$.

Proof. Let $y(t)$ be an arbitrary solution of system (1). Using the integral representation and Lemma 2.3 we have, for $t_0 > 0$,

$$\begin{aligned} y(t) &= X(t,t_0)y(t_0) + \int_{t_0}^{t} X(t,\tau)f(\tau,y(\tau))d\tau = \\ &= X(t,t_0)y(t_0) + \int_{t_0}^{t} X_1(t,\tau)f(\tau,y(\tau))d\tau + \int_{t_0}^{t} X_2(t,\tau)f(\tau,y(\tau))d\tau \\ &= X(t,t_0)y(t_0) + \int_{t_0}^{t} X_1(t,\tau)f(\tau,y(\tau))d\tau + \int_{t_0}^{\infty} X_2(t,\tau)f(\tau,y(\tau))d\tau \\ &\quad - \int_{t_0}^{\infty} X_2(t,\tau)f(\tau,y(\tau))d\tau \end{aligned}$$

$$= X(t,t_0)\left[y(t_0)+\int_{t_0}^{\infty} X_2(t_0,\tau)f(\tau,y(\tau))d\tau\right]+\int_{t_0}^{t} X_1(t,\tau)f(\tau,y(\tau))d\tau$$

$$-\int_{t_0}^{\infty} X_2(t,\tau)f(\tau,y(\tau))d\tau \tag{10}$$

The absolute convergence of the improper integrals in (10) follows from the estimates

$$\begin{aligned}\int_{t_0}^{\infty} \|X_2(t,\tau)\|\,|f(\tau,y(\tau))|\,d\tau &\le \int_{t_0}^{\infty} N_2 e^{-\nu_2(\tau-t_0)}\eta(\tau)\,|y(\tau)|\,d\tau \\ &\le N_2 a_2 \int_{t_0}^{\infty} e^{-\nu_2(\tau-t_0)}\eta(\tau)e^{a(\tau-t_0)}\,|y(t_0)|\,d\tau \\ &\le N_2 a_2\,|y(t_0)| \int_{t_0}^{\infty} e^{a\tau}\eta(\tau)d\tau < \infty\end{aligned}$$

which hold due to (6), Lemma 2.2, and the conditions C_2 and C_4. Note that condition (9) implies C_3.

The solutions $y(t)$ and $x(t)$ of systems (1) and (2) are uniquely defined by their initial conditions. Thus, for each solution $y(t)$ of system (1) with initial condition $y(t_0) = y_0$, we put into correspondence the solution $x(t)$ with initial condition $x(t_0) = x_0$ given by

$$x(t_0) = y(t_0) + \int_{t_0}^{\infty} X_2(t_0,\tau)f(\tau,y(\tau))d\tau \tag{11}$$

Let us show that the correspondence between solutions $y(t)$ and $x(t)$ given by (11) is one-to-one under the proper choice of t_0.

Denote $x_0 = x(t_0)$ and $y_0 = y(t_0)$. For every fixed t_0 the set

$$\left\{(t_0,y_0),\, y_0 \in \mathcal{R}^n\right\} \tag{12}$$

uniquely describes the set of solutions of (1) due to the existence and uniqueness theorems for ODEs. Similary, the set

$$\left\{(t_0,x_0),\, x_0 \in \mathcal{R}^n\right\} \tag{13}$$

uniquely describes the entire set of solutions of (2).

In our notation, the solution $y(\tau)$ in (11) is $y = y(\tau,t_0,y_0)$. Additionally, denote

$$\Phi(t_0,y_0) = \int_{t_0}^{\infty} X_2(t_0,\tau)f(\tau,y(\tau,t_0,y_0))d\tau$$

Then (11) has the form

$$x_0 = y_0 + \Phi(t_0, y_0) \tag{14}$$

Thus, in order to establish one-to-one correspondence between the solutions of (1) and (2), it suffices to show that (14) gives one-to-one correspondence between the sets (12) and (13) for some t_0.

Thus, we need to establish that for fixed t_0, and for every $x_0 \in \mathcal{R}^n$ the equation (14) can be uniquely solved for $y_0 \in \mathcal{R}^n$.

Rewrite equation (14) in the form $y_0 = x_0 - \Phi(t_0, y_0)$ and show that the map $x_0 - \Phi(t_0, y_0)$ is a contraction map in $\mathcal{R}^n$ for every $x_0 \in \mathcal{R}^n$ and for some t_0.

Indeed, for all y_0 and $y_1 \in \mathcal{R}^n$, we have

$$\begin{aligned} |x_0 - \Phi(t_0, y_0) - x_0 + \Phi(t_0, y_1)| &\leq |\Phi(t_0, y_1) - \Phi(t_0, y_0)| \\ &\leq \int_{t_0}^{\infty} \|X_2(t_0, \tau)\| \, \eta(\tau) \, |y(\tau, t_0, y_1) - y(\tau, t_0, y_0)| \, d\tau. \end{aligned} \tag{15}$$

But

$$y(\tau, t_0, y_1) = y_1 + \int_{t_0}^{\tau} A(s) y(s, t_0, y_1) ds + \int_{t_0}^{\tau} f(s, y(s, t_0, y_1)) ds,$$

and

$$y(\tau, t_0, y_0) = y_0 + \int_{t_0}^{\tau} A(s) y(s, t_0, y_0) ds + \int_{t_0}^{\tau} f(s, y(s, t_0, y_0)) ds,$$

Subtracting the second equation from the first equation, we get

$$\begin{aligned} |y(\tau, t_0, y_1) - y(\tau, t_0, y_0)| \;\; &\leq \;\; |y_1 - y_0| + | \int_{\tau}^{t} a \, |y(s, t_0, y_1) - y(s, t_0, y_0)| \, ds \\ &\quad + \int_{t_0}^{t} \eta(s) \, |y(s, t_0, y_1) \text{-} y(s, t_0, y_0))| \, ds \; | \end{aligned}$$

and thus, by Grownwall-Beilman Lemma, we have

$$|y(\tau, t_0, y_1) - y(\tau, t_0, y_0)| \leq |y_1 - y_0| \, e^{a((\tau - t_0) + \int_{t_0}^{\tau} \eta(s) ds}.$$

Using condition C_3, we get

$$|y(\tau, t_0, y_1) - y(\tau, t_0, y_0)| \leq |y_1 - y_0| \, e^{a(\tau - t_0) + a_1}. \tag{16}$$

Substituting (16) into (15), we get

$$\begin{aligned}|\Phi(t_0, y_1) - \Phi(t_0, y_0)| &\le \int_{t_0}^{\infty} \|X_2(t_0, \tau)\| \, \eta(\tau) \, |y_1 - y_0| \, e^{a(\tau - t_0) + a_1} d\tau \\ &\le N_2 e^{a_1} \int_{t_0}^{\infty} e^{-\nu_2(\tau - t_0)} e^{a(\tau - t_0)} \eta(\tau) \, |y_1 - y_0| \\ &\le N_2 e^{a_1} \int_{t_0}^{\infty} e^{a\tau} \eta(\tau) \, |y_1 - y_0| \, . \end{aligned} \tag{17}$$

By Equation (9), we can choose $t_0 > 0$ such that

$$N_2 e^{a_1} \int_{t_0}^{\infty} e^{a\tau} \eta(\tau) < 1 \tag{18}$$

Then from (17) and (18), it follows that $x_0 - \Phi(t_0, x_0)$ is a contraction mapping in $\mathcal{R}^n$, thus (14) has a unique solution in $\mathcal{R}^n$ for certain t_0 and for any $x_0 \in \mathcal{R}^n$. Therefore, the correspondence between the solutions of systems (1) and (2) given by (11) is one-to-one.

To complete the proof of the theorem, it remains to prove (3) for the corresponding solutions $x(t)$ and $y(t)$.

Since $x(t)$ takes the form

$$x(t) = X(t, t_0) x(t_0),$$

and $x(t_0)$ is defined by (11); from (10), we have

$$|x(t) - y(t)| \le \left| \int_{t_0}^{t} X_1(t, \tau) f(\tau, y(\tau)) d\tau \right| + \left| \int_{t_0}^{\infty} X_2(t, \tau) f(\tau, y(\tau)) d\tau \right| . \tag{19}$$

Now, let us show that both integrals approach to zero for $t \to 0$. By Equations (5) and (8), we estimate the first integral as follows:

$$\begin{aligned}\int_{t_0}^{t} X_1(t,\tau)f(\tau,y(\tau)d\tau &\leq \int_{t_0}^{t} \|X_1(t,\tau)\|\,\eta(\tau)\,|y(\tau)|\,d\tau \\ &\leq \int_{t_0}^{t} N_1 e^{-\nu_1(t-\tau)}\eta(\tau)a_2\,|y(t_0)|\,e^{a(\tau-t_0)}d\tau \\ &\leq N_1 a_2\,|y(t_0)| \int_{t_0}^{t} e^{-\nu_1(t-\tau)}\eta(\tau)e^{a\tau}d\tau \\ &= N_1 a_2\,|y(t_0)| \left[\int_{t_0}^{\frac{t}{2}} e^{-\nu_1(t-\tau)}\eta(\tau)e^{a\tau}d\tau + \int_{\frac{t}{2}}^{t} e^{-\nu_1(t-\tau)}\eta(\tau)e^{a\tau}d\tau\right]\end{aligned}$$

$$\leq N_1 a_2\,|y(t_0)| \left[e^{-\frac{t}{2}}\int_{t_0}^{\infty} \eta(\tau)e^{a\tau}d\tau + \int_{\frac{t}{2}}^{t} \eta(\tau)e^{a\tau}d\tau\right].$$

By Equation (9), it is obvious that the right hand side of last inequality approaches 0 as $t \to \infty$.

To estimate the second integral in (19), we have

$$\begin{aligned}\left|\int_{t}^{\infty} X_2(t,\tau)f(\tau,y(\tau))d\tau\right| &\leq \int_{t}^{\infty} \|X_2(t,\tau)\|\,\eta(\tau)a_2 e^{a(\tau-t_0)}d\tau \\ &\leq N_2 a_2 \int_{t}^{\infty} \eta(\tau)e^{a\tau}d\tau \to 0, \text{ as } t\to\infty.\end{aligned}$$

Thus $\lim_{t\to\infty} |x(t)-y(t)| = 0$, and the result follows. This completes the proof of the theorem. ∎

Let us now consider the following example.

Example 3.2. Consider the following system of type (2) in $\mathcal{R}^2$:

$$\left\{\begin{array}{l}\dfrac{dx_1}{dt} = -(\tanh t)x_1 \\ \dfrac{dx_2}{dt} = x_1 + (\tanh t)x_2\end{array}\right\} \tag{20}$$

Here the matrix $A(t)$ has the form

$$\begin{pmatrix} -\tanh t & 0 \\ 1 & \tanh t \end{pmatrix}$$

It is not difficult to see that system (20) is exponentially dichotomic on $\mathcal{R}$.

The corresponding projectors are

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly $\left\| \sup_{t\in\mathcal{R}} A(t) \right\| \leq \sqrt{3}$, Then, by Theorem 1, the following system

$$\begin{Bmatrix} \dfrac{dy_1}{dt} = -(\tanh t)y_1 + f_1(t)(\sin t)y_2 \\ \dfrac{dy_2}{dt} = y_1 + (\tanh t)y_2 + f_2(t)(\sin t)y_1 \end{Bmatrix}$$

is asymptotically equivalent to system (1) if

$$\int_0^\infty |f_i(t)|\, e^{\sqrt{3}t} dt < \infty, \text{ for } i = 1, 2.$$

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