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An Alternative Method for Evaluating the Determinant of a Square Matrix

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Abstract

In [1] a strange property of the determinant of minors of a matrix was discussed. In this paper, we show that evaluation of the determinant of any square matrix can be obtained using this property.

Keywords: Matrix, determinant and entrywise

Introduction

Let M be the matrix of the minors of a square matrix A of order n. For every square submatrix of order *k;* $M_k = (M_{ij})$ of M, the determinant of a square submatrix of order (n-k) of A is defined as

$$\delta_k = \left|(a_{pq})\right|,\ 1\leq p, q \leq n;\ \text{p} \neq i,\ \text{q} \neq j$$

With this notion, the relationship

$$\left|M_{ij}\right| = |A|^{k-1}\delta_k \quad (1)$$

was proved in [1]. The result is trivially true for $k = 1$ and it is also true for $n = k$.

A particular case of equation (1) where $k = 2$ gave the equation $|M_2| = \delta_2|A|$, so that

$$|A| = \frac{1}{\delta_2}|M_2| \text{ provided } \delta_2 \neq 0 \quad (2)$$

The expression in (2) provides an easy way of obtaining the determinant of A.

Evaluation of the determinant of 3 × 3 dimensional matrices

Given any 3 × 3 dimensional matrices, each M_{ij} is a 2 × 2 matrix and δ_2 is of order $(3-2)$ which is a scalar quantity. δ_2 is chosen arbitrarily so that $\delta_2 \neq 0$ and M_{ij} is calculated for the complementary row/column to the selected δ_2.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, and we select $\delta_2 = a_{31} \neq 0$ then $M_2 = \begin{pmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{pmatrix}$ so that

$$|M_2| = \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} = m_{12}\, m_{23} - m_{13}\, m_{22}$$

$$= (a_{21}\, a_{33} - a_{23}\ a_{31})\,(a_{11}\, a_{32} - a_{12}\ a_{31}) - (a_{21}\, a_{32} - a_{22}\ a_{31})\,(a_{11}\, a_{33} - a_{13}\ a_{31})$$

$$=(a_{11}\, a_{21}\, a_{32}\, a_{33} + a_{12}\, a_{23}\, a^2{}_{31} + a_{13}\, a_{21}\, a_{31}\ a_{32} + a_{11}\, a_{22}\, a_{33}\, a_{31}) -(a_{12}\, a_{21}\, a_{31}\, a_{33} + a_{11}\, a_{23}\, a_{31}\ a_{32} + a_{11}\, a_{21}\, a_{32}\, a_{33} + a_{13}\, a_{22}\, a^2{}_{31})$$

$$=a_{31}\,[(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31})]$$

Hence

$$|M_2| = a_{31}|A| \ or \ |A| = \frac{|M_2|}{a_{31}} \tag{3}$$

If δ_2 is replaced by a_{22} on the right hand side of equation (2) then

$$M_2 = \begin{pmatrix} a_{11}\, a_{22} - a_{12}\, a_{21} & a_{12}\, a_{23} - a_{13}\ a_{22} \\ a_{21}\, a_{32} - a_{22}\, a_{31} & a_{22}\, a_{33} - a_{23}\ a_{32} \end{pmatrix}$$

so that

$$|M_2| = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} \tag{4}$$

That is, each entry in M_2 is the determinant of each adjacent 2 x 2 submatrices of A.

Evaluation of the determinant of $n \times n$ dimensional matrices

For a matrix of higher order, a sequence of every overlapping submatrices of order 3 × 3 evaluated by the determinant of adjacent 2 × 2 submatrices

$$(M_2^*)_k\ k = (n-1), (n-2) \ldots 1 \tag{5}$$

and sequence of component divisors

$$(\delta_2)_k\ k = (n-2), (n-3) \ldots 1 \tag{6}$$

are obtained so that

$$(M_2)_k = \cdot \frac{(M_2^*)_k}{(\delta_2)_k}, k = (n-2), (n-3) \ldots 1,$$

where $\cdot -$ denotes division is done entrywise.

$$(M_2)_{n-1} = (M_2^*)_{n-1} \tag{7}$$

And $|A| = | \cdot \frac{(M_2^*)_1}{(\delta_2)_1}|$ provided $(\delta_2)_1 \neq 0$ and has no zero component.

It should be noted that the division on the right hand side of equation (7) are done component wise. If $(\delta_2)_k$ is zero, row/column be interchanged to obtain nonzero $(\delta_2)_k$.

Sample Examples

(i). Given $A = \begin{pmatrix} 1\ 2\ 3 \\ 4 - 5\ 6 \\ 7\ 8\ 9 \end{pmatrix}$, $|A|$ is calculated as follows:

Take

$(\delta_2)_1 = -5$ and $(M_2^*)_2 = \begin{pmatrix} \begin{vmatrix} 1\ 2 \\ 4\ -5 \end{vmatrix} & \begin{vmatrix} 2\ 3 \\ -5\ 6 \end{vmatrix} \\ \begin{vmatrix} 4\ -5 \\ 7\ 8 \end{vmatrix} & \begin{vmatrix} -5\ 6 \\ 8\ 9 \end{vmatrix} \end{pmatrix}$

So that

$|(M_2^*)_2| = \begin{vmatrix} -13\ 27 \\ 67\ -93 \end{vmatrix}$ and $(M_2^*)_1 = -600$

Hence $|A| = \cdot \frac{|(M_2^*)_2|}{(\delta_2)_1} = \frac{-600}{-5} = 120$

(ii). Let $B = \begin{pmatrix} 1\ 0\ 2\ 3 \\ 2\ 0\ 4\ 1 \\ -1\ 3\ 0\ 3 \\ 4\ 4\ 1\ 1 \end{pmatrix}$, $|B|$ is also evaluated as follows:

If $(\delta_2)_2$ is selected as $\begin{vmatrix} 0\ 4 \\ 3\ 0 \end{vmatrix}$, it contain zero components hence the following interchanges are made col <1, 2> and <3, 4> so that

$B^* = \begin{pmatrix} 0\ 1\ 3\ 2 \\ 0\ 2\ 1\ 4 \\ 3\ -1\ 3\ 0 \\ 4\ 4\ 1\ 1 \end{pmatrix}$,

it is noted that $|B| = |B^*|$, now let $(\delta_2)_2 = \begin{vmatrix} 2\ 1 \\ -1\ 3 \end{vmatrix}$ then $(\delta_2)_1 = 7$

$$(M_2^*)_3 = \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 3 & -1 \\ 4 & 4 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -5 & 10 \\ -6 & 7 & -12 \\ 16 & -13 & 3 \end{pmatrix}$$

$$|(M_2^*)_3| = \begin{vmatrix} 0 & -5 & 10 \\ -6 & 7 & -12 \\ 16 & -13 & 3 \end{vmatrix} = |(M_2)_3|$$

$$|(M_2^*)_2| = \begin{vmatrix} -30 & -10 \\ -34 & -135 \end{vmatrix},$$

and by dividing $|(M_2^*)_2|$ component wise by $(\delta_2)_2$ we obtained

$$(M_2)_2 = \begin{pmatrix} -15 & -10 \\ 34 & -45 \end{pmatrix}$$

Therefore, $|(M_2^*)_2| = \begin{vmatrix} -15 & -10 \\ 34 & -45 \end{vmatrix} = 1015$

Hence $(M_2^*)_1 = 1015$

And according to our formula,

$$|B| = \cdot \frac{(M_2^*)_1}{(\delta_2)_1},$$

$$= \frac{1015}{7} = 145$$

Hence, $\begin{vmatrix} 1 & 0 & 2 & 3 \\ 2 & 0 & 4 & 1 \\ -1 & 3 & 0 & 3 \\ 4 & 4 & 1 & 1 \end{vmatrix} = 145$

Remark

Manual evaluation of n x n matrices with $n \geq 5$ is very tedious but the method of successive reduction has reduced the rigour. The order of the matrix is successively reduced by evaluating the determinant of adjacent 2 x 2 submatrices until the determinant is obtained.

Reference

[1] Ajibade A. O., and Rashid M. A. (2007), A strange property of the determinant of minors, International Journal of Mathematical Education in Science and Technology, 38:6, 852 – 858.

[2] Kreyszig, E. (1999), Advanced Engineering Mathematics, 8th edition, New York, John Wiley and Sons Inc., 341–350.