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Quantum Algorithm for Clique Problem

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Abstract

Quantum algorithm for the clique problem and its example are reported. When k of n are forming the clique, these members are decided. A computational complexity of a classical computation is $n!/((n-k)!k!)[= N]$ times. In the quantum algorithm by using quantum phase inversion gates and quantum inversion about mean gates, its computational complexity is about $N^{1/2}$ times.

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1. Introduction

A quantum computer can process speedily by a parallel calculation that uses quantum entangled states. Deutsch-Jozsa's algorithm for the rapid solution [1–3], Shor's algorithm for the factorization [2–4], Grover's algorithms for the database search [2,5–9], Brassard-Hoyer-Tapp's algorithm for counting [10,11], and so on are known. However, in the clique problem [12], its quantum algorithm hasn't been known yet. Therefore, it is examined this time, its result is reported.

2. Clique problem

When k of n is forming the clique, these members are decided. A computational complexity of a classical computation is $n!/((n-k)!k!)$ times because a number of the combinations chooses k from n, and its order is unrelated.

3. Quantum algorithm

3.1. Premise

It is assumed that k of n are forming the clique. However, n and k are integers of 1 or more. The members of the clique must have a connection mutually. Moreover, connections of the members of the clique and nonmembers, and connections of the nonmembers and others are included within a range where a root of the problem isn't changed. Then, it is assumed that $x_{i,j}$ is 1 when i and j have a connection, and $x_{i,j}$ is 0 when they don't have it. However, i and j are integers with $0 \le i < j \le n-1$. Next, a situation division is done according to a value of k.

(I) At $k = 1$ or n : It isn't necessary to calculate because there is no clique in $k = 1$, and all persons are the members of the clique in $k = n$.

(II) At $2 \le k \le n/2$: When k are composing the clique, a value of the product of all combinations of x_{a_d,a_e} that d and e are integers with $0 \le d < e \le k-1$ becomes 1. However, a_d and a_e correspond to i and j respectively. In a word, the value of

$$x_{a_0,a_1}x_{a_0,a_2}\cdots x_{a_0,a_{k-1}}x_{a_1,a_2}x_{a_1,a_3}\cdots x_{a_1,a_{k-1}}\cdots x_{a_{k-2},a_{k-1}} \tag{3.1}$$

becomes 1.

(III) At $n/2 < k \le n-2$: A range of k' is $2 \le k' < n/2$ when it is assumed that k' is $n-k$. When the value of eq.(1) is 1 in the combination of k except it of k' nonmembers, k are the members of the clique. Therefore, this computational complexity is equivalent to it that exchanged k for k' in (II).

(IV) At $k = n-1$: When it is assumed that k' is $n-k$, k' is 1. Then, the value of eq.(1) in the combination of k except it of k' nonmembers is 1.

In addition, it assumes it as follows.

N: $N = n!/((n-k)!k!)$.
$|a>$, $|b>$, and $|c>$: Quantum registers. However, at $2 \le k \le n/2$, $|a>$ consists of the quantum registers $|a_0>, |a_1>, \ldots,$ and $|a_{k-1}>$, and at $n/2 < k \le n-1$ ($k' = k-n$, $1 \le k' < n/2$), it consists of the quantum registers $|a'_0>, |a'_1>, \ldots,$ and $|a'_{k'-1}>$.
α : An integer with $log_2 n \le \alpha$.
H : Hadamard gate [2,3].
(A) : A quantum gate. When a state of $|a>$ is n or more, $|b>$ becomes $|b+1>$, and when it is $n-1$ or less, $|b>$ isn't changed.
W_0 : $W_0 = 2^{\alpha k}$.
W_m : $W_m = n^{k-m} n!/((n-m)!m!)$. m is an integer from 1 to k. Especially, $W_1 = n^k$ and $W_m/W_{m+1} = (m+1)n/(n-m)$.
(PI) : A quantum phase inversion gate [2,5–9]. It inverses the phase of a quantum

register when its state is a target state.
(IM) : A quantum inversion about mean gate [2,5–9]. It inverses the probability amplitude of a quantum register to a mean value of the probability amplitude of the quantum registers.
(OB) : An observation gate.
(B) : A quantum gate. When a relation of states of $|a_u >$ and $|a_{u+1} >$ in $|a >$ is $|a_u >< |a_{u+1} >$, $|b >$ isn't changed. At $|a_u >> |a_{u+1} >$, $|b >$ becomes $|b+1 >$.
(C) : A quantum gate. At $2 \leq k \leq n-1$, the value of eq.(1) is substituted for $|c >$.

3.2. Case of $2 \leq k \leq n/2$

First of all, the procedure for $2 \leq k \leq n/2$ is shown. $|a >$, $|b >$, and $|c >$ are set $|0 >$. And, $\boxed{\text{H}}$ is made to act on each of quantum registers $|a_0 >, |a_1 >, \ldots,$ and $|a_{k-1} >$ in $|a >$. Therefore, the total of the actions becomes αk times. At the (A) gate, a state of $|a_0 >$ is compared with n. At $|a_0 >\geq n$, $|b >$ becomes $|b+1 >$. At $|a_0 >< n$, $|b >$ isn't changed. This action is repeated in $|a_u >$ that u is an integer from 1 to $k-1$. Therefore, the total of the actions in the (A) gate is k times. As a result, in $u = k-1$, when all states of $|a_u >$ are integers from 0 to $n-1$, the state of $|b >$ is 0, and when the state of $|a_u >$ has the 1 or more integers that are n or more, the state of $|b >$ becomes 1 or more. (PI) gates and (IM) gates act on $|b >$, where a target state is 0. The frequency is a minimum even number β_0 times that is $(W_0/W_1)^{1/2} = (2^\alpha/n)^{k/2} \leq \beta_0$ because the (PI) gate and the (IM) gate are a couple by two. Therefore, if β_0 is 4, the (IP) gate and the (IM) gate act 2 times respectively. After this, when only $|b >$ is observed by the (OB) gate, states of $|b >$ remain only 0, and the number of combinations of $|a >$ becomes $W_1 = n^k$.

At the (B) gate, states of $|a_0 >$ and $|a_1 >$ are compared. In case of $|a_0 >< |a_1 >$, $|b >$ isn't changed. In case of $|a_0 >\geq |a_1 >$, $|b >$ becomes $|b+1 >$. (PI) gates and (IM) gates act on $|b >$, where a target state is 0. The frequency is minimum even number β_1 times that is $(W_1/W_2)^{1/2} = (2n/(n-1))^{1/2} \leq \beta_1$. After this, when only $|b >$ is observed by the (OB) gate, states of $|b >$ remain only 0, and the number of combinations of $|a >$ becomes $W_2 = n^{k-2}n!/((n-2)!2!) = n^{k-1}(n-1)/2$. Similarly, when $|a_0 >$ and $|a_1 >$ are exchanged for $|a_u >$ and $|a_{u+1} >$ respectively, this is repeated in $u = 1, \ldots, k-2$ (u is an integer). In a word, at the (B) gate, states of $|a_u >$ and $|a_{u+1} >$ are compared. In case of $|a_u >< |a_{u+1} >$, $|b >$ isn't changed. In case of $|a_u >\geq |a_{u+1} >$, $|b >$ becomes $|b+1 >$. (PI) gates and (IM) gates act on $|b >$, where a target state is 0. The frequency is minimum even number β_{u+1} that is $(W_{u+1}/W_{u+2})^{1/2} = ((u+2)n/(n-(u+1)))^{1/2} \leq \beta_{u+1}$. After this, when only $|b >$ is observed by the (OB) gate, states of $|b >$ remain only 0, and the number of combinations of $|a >$ becomes $W_{u+2} = n^{k-(u+2)}n!/((n-(u+2))!(u+2)!)$. As a result, in $u = k-2$, all combinations with $|a_d >< |a_e >$ that d and e are integers with $0 \leq d < e \leq k-1$ remain, and the number becomes $W_k = n^{k-k}n!/((n-k)!k!) = n!/((n-k)!k!) = N$. The number of actions for $|a >$ by the (B) gate is $2(k-1)$ times, and the number of actions for $|b >$ by the (OB) gate is $k-1$ times.

At the (C) gate, the value of eq.(1) according to the combination of $|a >$ is substituted for $|c>$. The number to get states of $|a>$ is k times. The (PI) gate and the (IM) gate act on $|c>$, where a target state is 1. The frequency is about $N^{1/2}$ times that is the even number. Afterwards, when $|a>$ and $|c>$ are observed by the (OB) gate, states of the members of the clique and the calculation value of $|c>$ are obtained.

3.3. Case of $n/2 < k \leq n-2$

When the range of k is $n/2 < k \leq n-2$, it is assumed that k' is $n-k$. $|a>$ consists of $|a'_0>$, $|a'_1>, \ldots,$ and $|a'_{k'-1}>$. The value of eq.(1) of the combinations of k except those of k' is calculated at the (C) gate. At the last (OB) gate, states of the combination of k' and the state of the calculation value of $|c>$ are obtained. The combination of k except it of k' becomes it of the members of the clique that is requested.

3.4. Case of $k = n-1$ $(k' = n-k = 1)$

$|a>$, $|b>$, and $|c>$ are set $|0>$. And, $\boxed{\text{H}}$ is made to act on $|a'_0>$ in $|a>$. Therefore, the total number becomes α times.

At the (A) gate, a state of $|a'_0>$ is compared with n. At $|a'_0>\geq n$, $|b>$ becomes $|b+1>$. At $|a'_0>< n$, $|b>$ isn't changed. Then, the (PI) gate and the (IM) gate act on $|b>$, where a target state is 0. The frequency is minimum even number β_0 times that is $(W_0/W_1)^{1/2} = (2^\alpha/n)^{1/2} \leq \beta_0$. Then, when only $|b>$ is observed by the (OB) gate, states of $|b>$ remain only 0. The number of states of $|a>$ becomes $W_1 = n = N$. Here, the action of the (B) gate is unnecessary.

At the (C) gate, in the combination of the $(n-1)$ except the 1 nonmember by a state of $|a>$, the value of eq.(1) is substituted for $|c>$. The (PI) gate and the (IM) gate act on $|c>$, where a target state is 1. The frequency is about $n^{1/2} = N^{1/2}$ times that is the even number. Afterwards, when $|a>$ and $|c>$ are observed by the (OB) gate, a state of 1 nonmember and a state of $|c>$ are obtained. Therefore, the $n-1$ except the 1 nonmember are the members of the clique.

4. Numerical calculation

It describes as follows as examples in case of $k = 16$, 17, and 32 in $n = 33$. The cases of $k = 16$ and 17 become the maximum computational complexity in $n = 33$. And the case of $k = 32$ $(k' = n-k = 33-32 = 1)$ becomes the minimum computational complexity in $n = 33$.

4.1. Case of $k = 16$ in $n = 33$

16 of the $0th$, $3rd$, $5th$, $8th$, $10th$, $13th$, $14th$, $17th$, $19th$, $21st$, $22nd$, $24th$, $26th$, $28th$, $29th$, and $31st$ are the members of the clique among 33 from the $0th$ to the $32nd$. When i and j that are integers with $0 \leq i < j \leq 32$ have a connection, $x_{i,j}$ is 1, and when they don't have a connection, $x_{i,j}$ is 0. Connections other than these will not be thought for easiness now. Even if they exist in extent into which connections other than these don't change the root of the problem, the value of $x_{a_0,a_1}x_{a_0,a_2}\cdots x_{a_0,a_{15}}$

$x_{a_1,a_2}x_{a_1,a_3}\cdots x_{a_1,a_{15}}\cdots x_{a_{14},a_{15}}$ that includes such combinations becomes only 0.

Well, $|a>$, $|b>$, and $|c>$ are set $|0>$. And, $\boxed{\text{H}}$ is made to act on each of $|a_0>, |a_1>, \ldots,$ and $|a_{15}>$ in $|a>$. Therefore, the total of the operations is $\alpha k = 6 \times 16 = 96$ times because $log_2 33 \approx 5.05 \leq 6 = \alpha$.

At the (A) gate, a state of $|a_0>$ is compared with $n = 33$. At $|a_0> \geq 33$, $|b>$ becomes $|b+1>$. At $|a_0> < 33$, $|b>$ isn't changed. This action is repeated in $|a_u>$ that u is an integer from 0 to 15. Therefore, the total of the actions in the (A) gate is 16 times. As a result, in $u = 15$, when each state of $|a_u>$ is one of an integer from 0 to 32, the state of $|b>$ is 0, and when the state of $|a_u>$ has the 1 or more integers that are n or more, the state of $|b>$ becomes 1 or more. (PI) gates and (IM) gates act on $|b>$, where a target state is 0. The frequency is $(W_0/W_1)^{1/2} = (64/33)^{16/2} \approx 200 \leq 200 = \beta_0$ times. Therefore, the (PI) gate and the (IM) gate act 100 times respectively. After this, when only $|b>$ is observed by the (OB) gate, states of $|b>$ remain only 0, and the number of combinations of $|a>$ becomes $W_1 = 33^{16} \approx 1.98 \times 10^{24}$.

At the (B) gate, states of $|a_0>$ and $|a_1>$ are compared. In case of $|a_0> < |a_1>$, $|b>$ isn't changed. In case of $|a_0> \geq |a_1>$, $|b>$ becomes $|b+1>$. (PI) gates and (IM) gates act on $|b>$, where a target state is 0. The frequency is $(W_1/W_2)^{1/2} = (66/32)^{1/2} \approx 1.44 \leq 2 = \beta_1$ times. After this, when only $|b>$ is observed by the (OB) gate, states of $|b>$ remain only 0, and the number of combinations of $|a>$ becomes $W_2 = 33^{16-2}33!/((33-2)!2!) = 33^{15}32/2 \approx 9.59 \times 10^{23}$. Similarly, when $|a_0>$ and $|a_1>$ are exchanged for $|a_u>$ and $|a_{u+1}>$ respectively, this is repeated in $u = 1, \ldots, 14$ (u is an integer). In a word, at the (B) gate, states of $|a_u>$ and $|a_{u+1}>$ are compared. In cace of $|a_u> < |a_{u+1}>$, $|b>$ isn't changed. In case of $|a_u> \geq |a_{u+1}>$, $|b>$ becomes $|b+1>$. (PI) gates and (IM) gates act on $|b>$, where a target state is 0. The frequency is $(W_2/W_3)^{1/2} = (99/31)^{1/2} \approx 1.72 \leq 2 = \beta_2$, $(W_3/W_4)^{1/2} = (132/30)^{1/2} \approx 2.10 \leq 4 = \beta_3, \ldots, (W_{10}/W_{11})^{1/2} = (363/23)^{1/2} \approx 3.97 \leq 4 = \beta_{10}$, $(W_{11}/W_{12})^{1/2} = (396/22)^{1/2} \approx 4.24 \leq 6 = \beta_{11}, \ldots, (W_{15}/W_{16})^{1/2} = (528/18)^{1/2} \approx 5.42 \leq 6 = \beta_{15}$ times respectively.

After this, when only $|b>$ is observed by (OB) gate, states of $|b>$ remain only 0, and the number of combinations of $|a>$ becomes $W_3 = 33^{16-3}33!/(30!3!) = 33^{14}32 \cdot 31/6 \approx 3.00 \times 10^{23}$, $W_4 = 33^{16-4}33!/(29!4!) = 33^{13}32 \cdot 31 \cdot 30/24 \approx 6.83 \times 10^{22}, \ldots, W_{16} = 33^{16-16}33!/(17!16!) = 33!/(17!16!) \approx 1.17 \times 10^9$ respectively. The number of actions for $|a>$ by the (B) gate is 30 times, and the number of actions for $|b>$ by (B) gate is $k - 1 = 15$ times.

At the (C) gate, the value of $x_{a_0,a_1}x_{a_0,a_2}\cdots x_{a_0,a_{15}}x_{a_1,a_2}x_{a_1,a_3}\cdots x_{a_1,a_{15}}\cdots x_{a_{14},a_{15}}$ according to the combination of $|a>$ is substituted for $|c>$. The number to get states of $|a>$ is 16 times. The (PI) gate and the (IM) gate act on $|c>$, where a target state is 1. The frequency is $N^{1/2} \approx 34205.3 \leq 34206$ times. Afterwards, when $|a>$ and $|c>$ are observed by the (OB) gate, states of the members of the clique and the calculation value of $|c>$ are obtained. They are $0, 3, \ldots, 31$, and 1.

4.2. Case of $k = 17$ ($k' = 33 - 17 = 16$) in $n = 33$

17 except 16 members of the clique that shows with section 4.1 are made the members of a new clique. The quantum registers of $|a>$ that corresponds to $k' = 16$ are $|a'_0>$, $|a'_1>, \ldots,$ and $|a'_{15}>$. We calculate as well as section 4.1, and at the (C) gate, the value of $x_{a_0,a_1}x_{a_0,a_2}\cdots x_{a_0,a_{16}}x_{a_1,a_2}x_{a_1,a_3}\cdots x_{a_1,a_{16}}\cdots$
$x_{a_{15},a_{16}}$ except states of $|a'_0>, |a'_1>, \ldots,$ and $|a'_{15}>$ is substituted for $|c>$.

And, we calculate as well as section 4.1, and when $|a>$ and $|c>$ are observed by the (OB) gate, the combination of 16 nonmembers and a state of $|c>$ are obtained. They are $0, 3, \ldots, 31$, and 1. Therefore, 17 except 16 nonmembers of $0, 3, \ldots,$ and 31 become the members of the clique that is requested.

4.3. Case of $k = 32$ ($k' = 33 - 32 = 1$) in $n = 33$

It is assumed that the $7th$ person alone isn't a member of the clique among 33 from the $0th$ to the $32nd$. In $x_{i,j}$ that i and j are integers with $0 \le i < j \le 32$, when $7th$ isn't contained in i or j, $x_{i,j}$ is 1, and when $7th$ is contained in i or j, $x_{i,j}$ is 0. Well, $|a>$, $|b>$, and $|c>$ are set $|0>$. And, $\boxed{\mathrm{H}}$ is made to act on $|a'_0>$ in $|a>$. Therefore, the total number becomes $log_2 n = log_2 33 \approx 5.05 \le 6 = \alpha$ times.

At the (A) gate, a state of $|a'_0>$ is compared with $n = 33$. At $|a'_0> \ge 33$, $|b>$ becomes $|b+1>$. At $|a'_0> < 33$, $|b>$ isn't changed. Then, the (PI) gate and the (IM) gate act on $|b>$, where a target state is 0. The frequency is $(W_0/W_1)^{1/2} = (64/33)^{1/2} \approx 1.94 \le 2 = \beta_0$ times. Then, when only $|b>$ is observed by the (OB) gate, states of $|b>$ remain only 0. The number of states of $|a>$ becomes $W_1 = n = 33 = N$. Here, the action of the (B) gate is unnecessary.

Next, at the (C) gate, in the combination of 32 except the 1 nonmember by a state of $|a>$, the value of $x_{a_0,a_1}x_{a_0,a_2}\cdots x_{a_0,a_{31}}x_{a_1,a_2}x_{a_1,a_3}\cdots x_{a_1,a_{31}}\cdots x_{a_{30},a_{31}}$ is substituted for $|c>$. The (PI) gate and the (IM) gate act on $|c>$, where a target state is 1. The frequency is $n^{1/2} = N^{1/2} = 33^{1/2} \approx 5.74 \le 6$ times. Afterwards, when $|a>$ and $|c>$ are observed by the (OB) gate, states of the 1 nonmember and $|c>$ are obtained. They are 7 and 1 respectively. Therefore, 32 except the 1 nonmember as $7th$ are the members of the clique.

5. Discussion and Summary

The computational complexity of the quantum algorithm is generally as follows.

(I) For $2 \le k \le n/2$: In the order of the actions, frequencies are αk times at $\boxed{\mathrm{H}}$, k times at the (A) gates, about $(W_0/W_1)^{1/2} = (2^\alpha/n)^{k/2}$ times at (PI) gates and (IM) gates, once at the (OB) gate, $2(k-1)$ times at the (B) gates, about $\Sigma_{m=1}^{k-1}(W_m/W_{m+1})^{1/2} = \Sigma_{m=1}^{k-1}((m+1)n/(n+m))^{1/2}$ times at (PI) gates and (IM) gates, $k-1$ times at the (OB) gate, k times at the (C) gates, $(n!/((n-k)!k!))^{1/2} = N^{1/2}$ times at the (PI) gates and the (IM) gates, once at the (OB) gate. Therefore, total S_1 is $(\alpha+5)k - 1 + (2^\alpha/n)^{k/2} + \Sigma_{m=1}^{k-1}(W_m/W_{m+1})^{1/2} +$

$N^{1/2}$. S_1 is 34647 times in section 4.1 for $n = 33$ and $k = 16$. In the classical computation, N is about 1.17×10^9 times.

(II) For $n/2 < k \leq n - 2$ $(k' = n - k, 2 \leq k' < n/2)$: In the above-mentioned example, a frequency of case of k is same as that of k'. Therefore, in section 4.2 for $n = 33$ and $k = 17$ $(k' = 16)$, S_1 is 34647 times. In the classical computation, N is about 1.17×10^9 times.

(III) For $k = n - 1$ $(k' = n - k = 1)$: In the order of the actions, frequencies are α times at $\boxed{\text{H}}$, once at the (A) gate, about $(W_0/W_1)^{1/2} = (2^\alpha/n)^{1/2}$ times at the (PI) gates and the (IM) gates, once at the (OB) gate, once at the (C) gate, about $n^{1/2} = N^{1/2}$ times at the (PI) gates and the (IM) gates, once at the (OB) gate. Therefore, total S_2 is $\alpha + 4 + (2^\alpha/n)^{1/2} + N^{1/2}$. In section 4.3 for $n = 33$ and $k = 32$ $(k' = 1)$, S_2 is 18 times. In the classical computation, N is 33 times.

From the above-mentioned, both S_1 and S_2 are $N^{1/2}$ times when n is large enough. For instance, in a case of $n = 100$ and $k = 50$, the computational complexity of the quantum algorithm is $N^{1/2} \approx 3.18 \times 10^{14}$, and that of the classical computation is $N = 100!/(50!50!) \approx 1.01 \times 10^{29}$. Therefore, high-speed processing becomes possible by the quantum algorithm for the clique problem.

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