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On the Existence of Infinitely Many Primes of the Form x^2+1

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Abstract

In this paper we point out that very superior continuity of root Mersenne primes and very weak continuity of non-root Mersenne primes among known Mersenne primes may imply existence of infinitely many Mersenne primes and existence of infinitely many composite Fermat numbers and finitely many Fermat primes, from which we may infer every Fermat number is composite for $n>4$ to indicate existence of infinitely many primes of the form x^2+1.

Keywords: root Mersenne prime; composite Fermat number; Fermat prime; prime of the form x^2+1

Introduction

Our previous work[1] considered there exist 30 known root Mersenne primes i.e. M_2, M_3, M_5, M_7, M_{17}, M_{19}, M_{89}, M_{107}, M_{521}, M_{2203}, M_{4253}, M_{9689}, M_{9941}, M_{11213}, M_{19937}, M_{21701}, M_{86243}, M_{216091}, M_{756839}, M_{859433}, $M_{1257787}$, $M_{1398269}$, $M_{2976221}$, $M_{3021377}$, $M_{6972593}$, $M_{20996011}$, $M_{25964951}$, $M_{32582657}$, $M_{37156667}$, $M_{43112609}$ among 47 known Mersenne primes i.e. M_2, M_3, M_5, M_7, M_{13}, M_{17}, M_{19}, M_{31}, M_{61}, M_{89}, M_{107}, M_{127}, M_{521}, M_{607}, M_{1279}, M_{2203}, M_{2281}, M_{3217}, M_{4253}, M_{4423}, M_{9689}, M_{9941}, M_{11213}, M_{19937}, M_{21701}, M_{23209}, M_{44497}, M_{86243}, M_{110503}, M_{132049}, M_{216091}, M_{756839}, M_{859433}, $M_{1257787}$, $M_{1398269}$, $M_{2976221}$, $M_{3021377}$, $M_{6972593}$, $M_{13466917}$, $M_{20996011}$, $M_{24036583}$, $M_{25964951}$, $M_{30402457}$, $M_{32582657}$, $M_{37156667}$, $M_{42643801}$, $M_{43112609}$, which include M_p for p=2,3,5,7 and M_p to accord with $p–F_0$ divided by 8 or $p–F_1$ divided by 6, and advanced an accurate distribution law of root Mersenne primes, that is, there exist $t_{n+1}=2^n$ root Mersenne primes for $F_n–1<p<F_{n+1}–1$(n=0,1,2,3,…) and there exist $s_{n+1}=2^{n+1}$ root Mersenne primes for $p<F_{n+1}–1$, where F_n is Fermat number. It led to appearing of a description for distribution law of Mersenne primes by using number of root Mersenne primes i.e. there exist $s_{n+1}–1$ Mersenne primes for $F_n–1<p<F_{n+1}–1$(n=0,1,2,3,…) and there exist $2(s_{n+1}–1)–n$

Mersenne primes for $p<F_{n+1}-1$, and we verified the distribution law of root Mersenne primes and the description for distribution law of Mersenne primes accurately accord with real existence of known root Mersenne primes and Mersenne primes for $p<F_{3+1}-1=65536$. From the distribution of root Mersenne primes among known Mersenne primes we discover existence of very superior continuity of root Mersenne primes and very weak continuity of non-root Mersenne primes. It indicates all of Mersenne primes existing for $F_n-1<p<F_{n+1}-1$ will be root Mersenne primes when $n\geq n_0$ and there are infinitely many Mersenne primes if distribution law of root Mersenne primes is true, where n_0 is an integer greater than 4, so that we may infer our description for distribution law of Mersenne primes will be only an approximate law when F_{n+1} being composite Fermat number. These results may imply there exist infinitely many composite Fermat numbers and finitely many Fermat primes, and every Fermat number is composite for $n>4$, from which we may infer there are infinitely many primes of the form x^2+1.

Superior Continuity of Root Mersenne Primes

The distribution of known root Mersenne primes makes us discover existence of very superior continuity of root Mersenne primes among known Mersenne primes, that is, 1st to 4th Mersenne primes all are root Mersenne primes, 21st to 25th Mersenne primes all are root Mersenne primes and 31st to 38th Mersenne primes all are root Mersenne primes. It has been confirmed that there exist no any undiscovered Mersenne primes between the 1st and the 40th Mersenne primes[2], therefore such a superior continuity of known root Mersenne primes is true from the 1st Mersenne prime to the 40th Mersenne prime. But continuity of so-called non-root Mersenne primes (i.e. other Mersenne primes besides root Mersenne primes among Mersenne primes) is very weak from the 1st Mersenne prime to the 40th Mersenne prime, because we do not discover existence of continuous non-root Mersenne prime alignment whose number of non-root Mersenne primes is greater than 2 for 1st to 40th Mersenne primes. It seems to imply superior continuity of root Mersenne primes will be stronger and stronger with increase of p-value so that number of root Mersenne primes in continuous root Mersenne prime alignments whose number of root Mersenne primes is greater than 2 can be described as $Q_{m+1}=Q_m+3^{m-1}$(m=1,2,3,…), where $Q_1=4$ is number of root Mersenne primes (M_2, M_3, M_5, M_7) in the first continuous root Mersenne prime alignment with number of root Mersenne primes being greater than 2. From it we get $Q_{1+1}=Q_1+3^0=4+1=5$ being number of root Mersenne primes (M_{9689}, M_{9941}, M_{11213}, M_{19937}, M_{21701}) in the second continuous root Mersenne prime alignment with number of root Mersenne primes being greater than 2 and $Q_{2+1}=Q_2+3^1=5+3=8$ being number of root Mersenne primes (M_{216091}, M_{756839}, M_{859433}, $M_{1257787}$, $M_{1398269}$, $M_{2976221}$, $M_{3021377}$, $M_{6972593}$) in the third continuous root Mersenne prime alignment with number of root Mersenne primes being greater than 2. It shows that the formula is true for 1st to 40th known Mersenne primes.

From above formula we may get $Q_{3+1}=Q_3+3^2=8+9=17$, $Q_{4+1}=Q_4+3^3=17+27=44$, $Q_{5+1}=Q_5+3^4=44+81=125$, $Q_{6+1}=Q_6+3^5=125+243=368$, $Q_{7+1}=Q_7+3^6=368+729=1097$, $Q_{8+1}=Q_8+3^7=1097+2187=3284$ for m=3,4,5,6,7,8. If such a stronger and stronger continuity of root Mersenne primes and accurate distribution law of root Mersenne primes are true and we suppose that there exist no any continuous non-root Mersenne prime alignments whose number of non-root Mersenne primes is greater than 2 among all of Mersenne primes and there must exist one continuous root Mersenne prime alignment with number of root Mersenne primes being greater than 2 for F_n–$1<p<F_{n+1}–1$ when $n>2$ but all of Mersenne primes existing for $F_n–1<p<F_{n+1}–1$ will be root Mersenne primes to lead number of Mersenne primes existing for $F_n–1<p<F_{n+1}–1$ to be 2^n if $Q_{m+1}=Q_m+3^{m-1}\geq t_{n+1}=2^n$ for $F_n–1<p<F_{n+1}–1$ when $n>2$, then we may expect all of Mersenne primes existing for $F_n–1<p<F_{n+1}–1$ will be root Mersenne primes when $n\geq n_0$, and we may infer $4<n_0<n_{max}$, where n_{max} is n-value of the largest composite Fermat number in the first continuous known composite Fermat number alignment and we know $n_{max}=32$ to this day[3]. From it we may get $t_{3+1}=8$ and $Q_{1+1}=5$ for $F_3–1<p<F_{3+1}–1$, $t_{4+1}=16$ and $Q_{2+1}=8$ for $F_4–1<p<F_{4+1}–1$, $t_{5+1}=32$ and $Q_{3+1}=17$ for $F_5–1<p<F_{5+1}–1$, $t_{6+1}=64$ and $Q_{4+1}=44$ for $F_6–1<p<F_{6+1}–1$, $t_{7+1}=128$ and $Q_{5+1}=125$ for $F_7–1<p<F_{7+1}–1$ but $t_{8+1}=256$ and $Q_{6+1}=368$ for $F_8–1<p<F_{8+1}–1$, $t_{9+1}=512$ and $Q_{7+1}=1097$ for $F_9–1<p<F_{9+1}–1$, $t_{10+1}=1024$ and $Q_{8+1}=3284$ for $F_{10}–1<p<F_{10+1}–1$. If these results are true then we may expect $n_0=8$ because of $Q_{m+1}=Q_m+3^{m-1}>t_{n+1}=2^n$ for $F_n–1<p<F_{n+1}–1$ when $n\geq n_0=8$, so that all of Mersenne primes existing for F_n–$1<p<F_{n+1}–1$ will be root Mersenne primes when $n\geq n_0=8$ from our advanced supposition. It means that there will exist no any non-root Mersenne primes for F_n–$1<p<F_{n+1}–1$ when $n\geq n_0=8$. If distribution law of root Mersenne primes for F_n–$1<p<F_{n+1}–1$ is true then we may infer there are infinitely many root Mersenne primes and finitely many non-root Mersenne primes from our advanced supposition. It implies that there may exist infinitely many Mersenne primes because root Mersenne prime itself is also Mersenne prime.

Approximate Character of Description for Distribution Law of Mersenne Primes

If it is true that $n_0=8$ then we may infer number of Mersenne primes existing for F_n–$1<p<F_{n+1}–1$ when $n_0\leq n<n_{max}$ is 2^n being a composite number from accurate distribution law of root Mersenne primes for $F_n–1<p<F_{n+1}–1$ and all of Mersenne primes existing for $F_n–1<p<F_{n+1}–1$ to be root Mersenne primes when $n\geq n_0$ but F_{n+1} is a known composite Fermat number when $n_0\leq n<n_{max}$. From it we can advance following conjecture.

Conjecture 1: If F_{n+1} is composite Fermat number for $n\geq 4$, then number of Mersenne primes existing for $F_n–1<p<F_{n+1}–1$ is a composite number. If number of Mersenne primes existing for $F_n–1<p<F_{n+1}–1$ is a composite number for $n\geq 4$, then F_{n+1} is composite Fermat number.

The conjecture makes us feel our previous description for distribution law of Mersenne primes by using number of root Mersenne primes is accurately tenable only for $p<F_{3+1}–1=65536$, which means the description for distribution law of Mersenne primes is an accurate law when F_{n+1} being Fermat prime but may be only an approximate law when F_{n+1} being composite Fermat number.

From the conjecture we understand that number of Mersenne primes existing for F_4–1<p<F_{4+1}–1 can not be prime because F_{4+1}=4294967297 is the first known composite Fermat number as Euler showed that F_5=4294967297=641×6700417 in 1732[3], however, in our previous work we expected that number of Mersenne primes existing for F_4–1<p<F_{4+1}–1 will be 31 being prime from our description for distribution law of Mersenne primes, so that the description for distribution law of Mersenne primes must be contradictory to Conjecture 1. According to the conjecture, we should expect that number of Mersenne primes existing for F_4–1<p<F_{4+1}–1 will not be 31 but a composite number. In our previous work we expected that number of undiscovered root Mersenne primes existing for F_4–1<p<F_{4+1}–1 is 16–14=2 because number of root Mersenne primes calculated by distribution law of root Mersenne primes is 2^4=16 but number of known root Mersenne primes is 14 for F_4–1<p<F_{4+1}–1, and number of undiscovered non-root Mersenne primes existing for F_4–1<p<F_{4+1}–1 is 11–2=9, however, we may get a new result from our advanced supposition and Conjecture 1. In this paper we suppose that there exist no any continuous non-root Mersenne prime alignments whose number of non-root Mersenne primes is greater than 2 among all of Mersenne primes so that we may infer there exist at most 6 non-root Mersenne primes for 43112610<p<4294967296 if there exist no any undiscovered Mersenne primes between the 40th and the 47th Mersenne primes and distribution law of root Mersenne primes is true. From it we may infer there are at most 28 Mersenne primes for 65536<p<4294967296 because number of known Mersenne primes is 20 for 65536<p<4294967296, which means number of Mersenne primes existing for F_4–1<p<F_{4+1}–1 will not be 31 but a smaller composite number than 31.

Fermat Numbers and Primes of The Form x^2+1

As we have discussed, if there generally exist stronger and stronger continuity of root Mersenne primes and very weak continuity of non-root Mersenne primes among all of Mersenne primes then we may expect that all of Mersenne primes existing for F_n–1<p<F_{n+1}–1 will be root Mersenne primes when n≥n_0=8, which means there will exist no any non-root Mersenne primes for F_n–1<p<F_{n+1}–1 to lead number of Mersenne primes existing for F_n–1<p<F_{n+1}–1 to be 2^n when n≥n_0=8. If distribution law of root Mersenne primes for F_n–1<p<F_{n+1}–1 and Conjecture 1 are true and it is true that all of Mersenne primes existing for F_n–1<p<F_{n+1}–1 will be root Mersenne primes when n≥n_0 then we may infer there are infinitely many composite Fermat numbers and finitely many Fermat primes because 2^n to be number of Mersenne primes existing for F_n–1<p<F_{n+1}–1 will be always composite numbers when n≥n_0 to indicate F_{n+1} will be always composite Fermat numbers when n≥n_0 according to Conjecture 1. From expecting existence of n_0=8<n_{max}=32 we may infer that every Fermat number is composite for n>4. Considering existence of recurrence relation between two neighbor Fermat numbers i.e. F_{n+1}=(F_n–1)^2+1[3] and Fermat numbers F_{n+1}=(F_n–1)^2+1 to be a special case of numbers of the form x^2+1, we may infer there are infinitely many composite numbers of the form x^2+1 from existence of infinitely many composite Fermat numbers but we can not infer there are finitely many primes of the form x^2+1 from existence of finitely many Fermat primes.

If it is true that every Fermat number is composite for $n>4$ then we may infer there are infinitely many prime factors of Fermat numbers to be able to be written in the form $k\cdot 2^a+1$, where k is odd positive integer but a is positive integer. However, existence of infinitely many prime factors $(k^{1/2}\cdot 2^{a/2})^2+1$ of Fermat numbers among all of prime factors of Fermat numbers is very important to indicate existence of infinitely many primes of the form x^2+1, so that we should advance following conjecture.

Conjecture 2: There is at least one prime factor $(k^{1/2}\cdot 2^{a/2})^2+1$ of Fermat number for $F_n-1\le a<F_{n+1}-1$ (n=0,1,2,3,…), where $k^{1/2}$ is odd positive integer and a is even positive integer.

In fact, we have known there exists one prime factor $(1^{1/2}\cdot 2^{2/2})^2+1=5$ of Fermat number F_1 to be prime of the form $(k^{1/2}\cdot 2^{a/2})^2+1$ for $F_0-1\le a<F_{0+1}-1$ i.e. $2\le a<4$; there exist two prime factors $(1^{1/2}\cdot 2^{4/2})^2+1=17$ and $(1^{1/2}\cdot 2^{8/2})^2+1=257$ of Fermat numbers F_2, F_3 to be primes of the form $(k^{1/2}\cdot 2^{a/2})^2+1$ for $F_1-1\le a<F_{1+1}-1$ i.e. $4\le a<16$; there exists one prime factor $(1^{1/2}\cdot 2^{16/2})^2+1=65537$ of Fermat number F_4 to be prime of the form $(k^{1/2}\cdot 2^{a/2})^2+1$ for $F_2-1\le a<F_{2+1}-1$ i.e. $16\le a<256$; there exists one prime factor $169\cdot 2^{63686}+1=(13\cdot 2^{31843})^2+1$ of Fermat number F_{63679} to be prime of the form $(k^{1/2}\cdot 2^{a/2})^2+1$ for $F_3-1\le a<F_{3+1}-1$ i.e. $256\le a<65536$ (it was discovered by H. Dubner on 19 May 1998[4]); and there exists one prime factor $25\cdot 2^{2141884}+1=(5\cdot 2^{1070942})^2+1$ of Fermat number $F_{2141872}$ to be prime of the form $(k^{1/2}\cdot 2^{a/2})^2+1$ for $F_4-1\le a<F_{4+1}-1$ i.e. $65536\le a<4294967296$ (it was discovered by G. Granowski on 9 Sep 2011[4]).

From Conjecture 2 we can get its one corollary.

Corollary 1: If Conjecture 2 is true, then there are infinitely many prime factors $(k^{1/2}\cdot 2^{a/2})^2+1$ of Fermat numbers.

Proof: Take $n\to\infty$ in Conjecture 2 then we will get the result.

If Corollary 1 is true then we can get another corollary of Conjecture 2.

Corollary 2: If Corollary 1 is true, then there are infinitely many primes of the form x^2+1.

Proof: Because prime factors $(k^{1/2}\cdot 2^{a/2})^2+1$ of Fermat numbers are a special case of primes of the form x^2+1, we will get the result.

From it we see Conjecture 2 and its two corollaries may show whether there are infinitely many primes of the form x^2+1, which is one of four basic problems about primes mentioned by Landau in 1912[5].

Conclusion

From expecting general existence of stronger and stronger continuity of root Mersenne primes and general existence of very weak continuity of non-root Mersenne primes among all of Mersenne primes we may infer that all of Mersenne primes existing for $F_n-1<p<F_{n+1}-1$ will be root Mersenne primes when $n \geq n_0=8$ to indicate existence of infinitely many Mersenne primes. If Conjecture 1 is true then we may infer our previous description for distribution law of Mersenne primes by using number of root Mersenne primes will be only an approximate law when F_{n+1} being composite Fermat number. If distribution law of root Mersenne primes for $F_n-1<p<F_{n+1}-1$ and Conjecture 1 are true then we may infer there are infinitely many composite Fermat numbers and finitely many Fermat primes, from which we may infer every Fermat number is composite for $n>4$ because of expecting existence of $n_0=8<n_{max}=32$. If it is true that every Fermat number is composite for $n>4$ then we may infer there are infinitely many primes of the form x^2+1 from Conjecture 2 and its two corollaries.

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