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Numerical Solutions to the Second Order Fredholm Integro-Differential Equations using the Spline Functions Expansion

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Abstract

In this Letter, we introduce a new technique to find an approximate solution for second order Fredholm integro-differential equations (FIDEs). This technique depends on approximate the solution using the spline functions expansion. Special attention is given to study the error estimation and the convergence of the proposed method. Also, the stability of the technique is presented. The numerical results are compared with the conventional approximate method, variational iteration method.

Keywords: Spline functions expansion; Variational iteration method; Fredholm integro-differential equations; Error estimation equations; Stability.

Introduction

Consider the following second order Fredholm integro-differential equation:

$$y''(x) = f(x, y(x), y'(x), \int_a^b K(x,t, y(t), y'(t))\, dt, \qquad a \le x \le b, \tag{1}$$

with the following initial conditions:

$$y(x_0) = c_1, \qquad y'(x_0) = c_2, \tag{2}$$

where f is given function and y is the unknown function to be found in the interval $[a,b]$. In [1] and [6] the authors introduced a method is an one-step method $o(h^{m+\alpha})$ in $y^{(i)}(x)$, $i=0,1,2$. Assuming that $f \in C[a,b]\times R^4$, $0<\alpha\leq 1$ and m is an arbitrary positive integer which is the number of iterations used in computing the spline functions defined in the method ([5]-[8]).

The rest of this paper is organized as follows: Section 2 is assigned to introduce some assumptions and procedure of the proposed method. In section 3, the error estimation and convergence are given. In section 4, the stability of the method is presented. In section 5, an example is solved by the proposed method, to illustrate and show the efficiency of the method. Also, the conclusions and remarks will appear in section 6.

Assumptions and procedure solution

We write (1) in the following form

$$y''(x)=f(x,y(x),y'(x),z(x)), \qquad a\leq x\leq b, \tag{3}$$

where

$$z(x)=\int_a^b K(x,t,y(t),y'(t))\,dt,$$

$$y(x_0)=c_1, \quad y'(x_0)=c_2.$$

Suppose that the function $f:[a,b]\times R^4\to R$ is continuous and satisfies the Lipschitz condition:

$$|f(x,y_1,v_1,z_1)-f(x,y_2,v_2,z_2)|\leq L_1\left[|y_1-y_2|+|v_1-v_2|+|z_1-z_2|\right], \tag{4}$$

for any (x,y_1,v_1,z_1) and (x,y_2,v_2,z_2) in the domain of definition of the function f.

Also, assume that the kernal $K:[a,b]\times[a,b]\times R^2\to R$ is a smooth bounded function and satisfies the Lipschitz condition [4]:

$$|K(x,t,y_1,v_1)-K(x,t,y_2,v_2)|\leq L_2[\,|y_1-y_2|+|v_1-v_2|\,], \tag{5}$$

for any (x,t,y_1,v_1) and (x,t,y_2,v_2) in the domain of definition of the kernal K. These conditions assure the existence of the unique solution of problem (1).

Let Δ be an uniform partition of the interval $[a,b]$ defined by the nodes:

$$\Delta := a=x_0<x_1<x_2<\ldots<x_k<x_{k+1}<\ldots<x_n:=b,$$

where $x_k=x_0+kh$, $h=\frac{b-a}{n}<1$ and $k=0,1,2,\ldots,n-1$.

Assume that the function y'' has a modulus of continuity:

$$w(y'', h) = w(h) = o(h^{\alpha}), \qquad 0 < \alpha \le 1.$$

Choosing the required positive integer m, then for any $[x_k, x_{k+1}]$, $k = 0,1,2,...,n-1$. We define the spline function approximating the solution y(x) by $S_{\Delta}(x)$ where:

$$S_{\Delta}(x) = S_k^m(x) = S_{k-1}^m(x_k) + S'^{m}_{k-1}(x_k)(x-x_k) + \int_{x_k}^{x}\int_{x_k}^{t} f(u, S_k^{m-1}(u), S'^{m-1}_k(u), Z_k^{m-1}(u))\, du dt, \tag{6}$$

where $Z_k^{m-1}(u) = \int_a^b K(u, t, S_k^{m-1}(u), S'^{m-1}_k(u))\, du$,

$$S_{-1}^m(x_0) = c_1, \quad S'^{m}_{-1}(x_0) = c_2, \quad S_{-1}^m(t) = c_1, \quad S'^{m}_{-1}(t) = c_2.$$

In Eq.(6) we use the following m iterations for

$x \in [x_k, x_{k+1}]$, $k = 0,1,2,...,n-1$, $j = 1,2,...,m$.

$$S_k^j(x) = S_{k-1}^m(x_k) + S'^{m}_{k-1}(x_k)(x - x_k) + \int_{x_k}^{x}\int_{x_k}^{t} f(u, S_k^{j-1}(u), S'^{j-1}_k(u), Z_k^{j-1}(u))\, du dt, \tag{7}$$

where $Z_k^{j-1}(u) = \int_a^b K(u, t, S_k^{j-1}(u), S'^{j-1}_k(u))\, du$,

$$S_k^0(x) = S_{k-1}^m(x_k) + S'^{m}_{k-1}(x_k)(x - x_k) + \frac{M_k}{2}(x - x_k)^2, \tag{8}$$

$$M_k = f(x_k, S_{k-1}^m(x_k), S'^{m}_{k-1}(x_k), \int_a^b K(x_k, t, S_{k-1}^m(t), S'^{m}_{k-1}(t))\, dt). \tag{9}$$

The Eqs.(7)-(9) present the main scheme which produced from the proposed method. From this scheme, we can obtain the approximate solution of the problem (1).

Error estimation and convergence

To estimate the error, it is convenient to represent the exact solution y(x) in various forms as described by the following scheme:

$$y^{0}(x) = y(x) = y(x_k) + y'(x_k)(x - x_k) + \frac{y''(\xi_k)}{2}(x - x_k)^2, \tag{10}$$
$$y(x_k) = y_k, \quad y'(x_k) = y'_k, \quad \xi_k \in (x_k, x_{k+1}).$$

For $i = 1,2,...,m$ we write:

$$y^{i}(x) = y(x) = y(x_k) + y'(x_k)(x - x_k) + \int_{x_k}^{x}\int_{x_k}^{t} f(u, t, y^{i-1}(u), y'^{\,i-1}(u), Z^{i-1}(u))\, du dt, \tag{11}$$

where $Z^{i-1}(u) = \int_a^b K(u, t, y^{i-1}(u), y'^{\,i-1}(u))\, du.$

Moreover, we denote to the estimated error of $y^{(i)}(x)$ at any point $x \in [a,b]$ where $i = 0,1,2$ by:

$$e(x) = | y(x) - S_\Delta(x)|, \qquad e_k = | y_k - S_\Delta(x_k)|, \tag{12}$$
$$e'(x) = | y'(x) - S'_\Delta(x)|, \qquad e'_k = | y'_k - S'_\Delta(x_k)|. \tag{13}$$

Lemma 1

Let α and β be non-negative real numbers and $\{A_i\}_{i=0}^{m}$ be a sequence satisfying $A_i \le \alpha + \beta A_{i+1}$ for $i = 1,2,...,m-1$, then:

$$A_1 \le \beta^{m-1} A_m + \alpha \sum_{i=0}^{m-2} \beta^i.$$

Lemma 2

Let α and β be non-negative real numbers, $\beta \ne 1$ and $\{A_i\}_{i=0}^{k}$ be a sequence satisfying $A_0 \ge 0$ and $A_{i+1} \le \alpha + \beta A_i$ for $i = 0,1,2,...,k$, then:

$$A_{k+1} \le \beta^{k+1} A_0 + \alpha \left[\frac{\beta^{k+1} - 1}{\beta - 1}\right].$$

Definition 1

For any $u \in [x_k, x_{k+1}]$, $k = 0,1,2,...,n-1$, and $j = 1,2,...,m$ we define the operator $T_{kj}(u)$ by:

$$T_{kj}(u) = | y^{m-j}(u) - S_k^{m-j}(u) | + | y'^{m-j}(u) - S'^{m-j}_k(u) |,$$

whose norm is defined by:

$$\| T_{kj} \| = \max_{u \in [x_k, x_{k+1}]} \{T_{kj}(u)\}.$$

Lemma 3

For any $u \in [x_k, x_{k+1}]$, $k = 0,1,2,...,n-1$, and $j = 1,2,...,m$

$$\| T_{km} \| \le (1 + \frac{3}{2} b_0) e_k + (2 + \frac{3}{2} b_0) e'_k + \frac{3}{2} h\, w(h), \tag{14}$$

$$\| T_{k1} \| \le b_1 e_k + b_2 e'_k + C_1 h^m\, w(h). \tag{15}$$

Proof

Using (4), (5), (8), (10), (12) and (13), we get:

$$|y^0(x) - S_k^0(x)| \le |y_k - S^m_{k-1}(x_k)| + |y'_k - S'^m_{k-1}(x_k)|$$
$$|x - x_k| + \frac{1}{2} |y''(\xi_k) - M_k| . |x - x_k|^2. \tag{16}$$

Since:

$$| y''_k - M_k | \le | y''(\xi_k) - y''_k | + | y''_k - M_k |$$
$$\le w(h) + L_1 \Big[| y_k - S^m_{k-1}(x_k) | + | y'_k - S'^m_{k-1}(x_k) | \Big] + L_2 [\int_a^b (| y(t) - S^m_{k-1}(t) |$$
$$+ | y'(t) - S'^m_{k-1}(t) |) \, dt].$$

But for $t \in [x_k, x_{k+1}]$, $e(t) = | y(t) - S^m_{k-1}(t) | \to e_k$ and

$e'(t) = | y'(t) - S'^m_{k-1}(t) | \to e'_k$ as $t \to x_k$.

Hence

$$|y''(\xi_k) - M_k| \le w(h) + L_1(e_k + e'_k) + L_2(b-a)(e_k + e'_k)$$
$$= w(h) + b_0(e_k + e'_k), \tag{17}$$

where $b_0 = L_1 + L_1 L_2 (b-a)$ is a constant independent of h.

Using (17) in (16) we get:

$$|\, y^{0}(x) - S_{k}^{0}(x)\,| \le e_{k} + h\,e'_{k} + \frac{h^{2}}{2}[\,w(h) + b_{0}(e_{k} + e'_{k})\,]$$
$$\le e_{k} + e'_{k} + \frac{h}{2}[\,w(h) + b_{0}(e_{k} + e'_{k})\,], \qquad (h < 1). \tag{18}$$

Similarly

$$|\, y'^{0}(x) - S'^{0}_{k}(x)\,| \le e'_{k} + h\,[\,w(h) + b_{0}(e_{k} + e'_{k})\,]. \tag{19}$$

Adding (18) and (19) we get:

$$\| T_{km} \| = \max_{x \in [x_{k}, x_{k+1}]} \{T_{km}(x)\} \le (1 + \frac{3}{2} b_{0}) e_{k} + (2 + \frac{3}{2} b_{0}) e'_{k} + \frac{3}{2} h\; w(h).$$

To prove (15), we compute $\| T_{km} \|$ and using (4), (5), (7), (11), (12) and (13)

$$|\, y^{m-j}(x) - S_{k}^{m-j}(x)\,| \le e_{k} + he'_{k} + L_{1} \int_{x_{k}}^{x} \int_{x_{k}}^{t} \left[T_{k(j+1)}(x)dx + L_{2} \int_{a}^{b} T_{k(j+1)}(t) \right] dt,$$

$$\max_{x \in [x_{k}, x_{k+1}]} |y^{m-j}(x) - S_{k}^{m-j}(x)| \le e_{k} + he'_{k} + \frac{h^{2}}{2} b_{0}$$
$$\|T_{k(j+1)}\|, \qquad (h < 1). \tag{20}$$

Similarly:

$$\max_{x \in [x_{k}, x_{k+1}]} |\, y'^{m-j}(x) - S'^{m-j}_{k}(x)\,| \le e'_{k} + hb_{0} \| T_{k(j+1)} \|. \tag{21}$$

Adding (20) and (21), we obtain:

$$\| T_{kj} \| \le e_{k} + 2e'_{k} + \frac{3}{2} b_{0} h \| T_{k(j+1)} \|.$$

Using Lemma 1, and the inequality (14) we get:

$$\| T_{k1} \| \le \left(\frac{3}{2} b_{0} h\right)^{m-1} \| T_{km} \| + (e_{k} + 2e'_{k}) \sum_{i=0}^{m-2} \left(\frac{3}{2} b_{0} h\right)^{i}$$
$$\le \left(\frac{3}{2} b_{0} h\right)^{m-1} \left[(1 + \frac{3}{2} b_{0}) e_{k} + (2 + \frac{3}{2} b_{0}) e'_{k} + \frac{3}{2} h w(h) \right] + (e_{k} + 2e'_{k}) \sum_{i=0}^{m-2} \left(\frac{3}{2} b_{0} h\right)^{i}$$
$$\le b_{1} e_{k} + b_{2} e'_{k} + c_{1} h^{m} w(h),$$

where $b_1 = \sum_{i=0}^{m} \left(\frac{3}{2} b_0\right)^i$, $b_2 = 2 \sum_{i=0}^{m-1} \left(\frac{3}{2} b_0\right)^i$ and $c_1 = \frac{3}{2}\left(\frac{3}{2} b_0\right)^{m-1}$, are constants independent of h.

Lemma 4

Let $e(x)$ and $e'(x)$ be defined as in (12), (13), then there exist constants b_3, b_4, b_5, b_6, c_2 and c_3 independent of h such that the following inequalities hold:

$$e(x) \le (1 + hb_3)e_k + hb_4 e'_k + c_2 h^{m+2} w(h),$$

$$e'(x) \le hb_5 e_k + (1 + hb_6)e'_k + c_3 h^{m+1} w(h),$$

where $b_3 = \frac{b_0 b_1}{2}$, $b_4 = 1 + \frac{b_0 b_2}{2}$, $b_5 = b_0 b_1$, $b_6 = b_0 b_2$, $c_2 = \frac{b_0 c_1}{2}$, and $c_3 = b_0 c_1$.

Proof

Using (4), (5), (6), (11), (12), (13) and (15) it is easy to prove the lemma.

Definition 2

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order, then we say that $A \le B$ iff:

[i.] both a_{ij} and b_{ij} are non-negatives;

[ii.] $a_{ij} \le b_{ij}$, $\forall i, j$.

In view of this definition and using the matrix notation:

$$E(x) = (e(x)\, e'(x))^T, \quad E == (e_k e'_k)^T \text{ and } C = (c_2 c_3)^T,$$

where T stands for the transpose, then from Lemma 4, we can write:

$$E(x) = (I + hA)E_k + Ch^{m+1} w(h), \quad (22)$$

where I is unit matrix and $A = \begin{pmatrix} b_3 & b_4 \\ b_5 & b_6 \end{pmatrix}$.

Definition 3
Let $T=[t_{ij}]$ be a $m\times n$ matrix then, we define:

$$\| T \| = \max_i \sum_{j=0}^{n} | t_{ij} |.$$

Using this definition, the inequality (22) yields:

$$\| E(x) \| \le (1+h\| A \|)\| E_k \| + \| C \| h^{m+1} w(h).$$

This inequality holds for $x\in[a,b]$. Setting $x=x_{k+1}$, we obtain:

$$\| E_{k+1} \| \le (1+h\| A \|)\| E_k \| + \| C \| h^{m+1} w(h).$$

Using Lemma 2, and noting that $\| E_0 \| = 0$ we get:

$$\| E(x) \| \le b_7 h^m w(h),$$

where $b_7=\frac{\| C \|}{\| A \|}[e^{\| A \|(b-a)}-1]$ is a constant independent of h. Using Definition 3, we get:

$$e(x) \le b_7 h^m w(h), \tag{23}$$

$$e'(x) \le b_7 h^m w(h). \tag{24}$$

Now we are going to estimate $| y''(x) - S''_\Delta(x) |$. Using (4), (5), (6), (11), (12), (13), (15), (23), and (24), we get:

$$| y''(x) - S''_\Delta(x) | \le b_8 h^m w(h),$$

where $b_8=b_0[b_7(b_1+b_2)+c_1]$ is a constant independent of h. Hence from above Lemma we have arrive to the following theorem.

Theorem 1
Let $y(x)$ be the exact solution of the problem (1), $S_\Delta(x)$ given by (6) is the approximate solution for the problem, $f\in C[a,b]\times R^4$, then the following inequalities

$$| y^{(p)}(x) - S_\Delta^{(p)}(x) | \le b_9 h^m w(h),$$

hold for $x\in[a,b]$and $p=0,1,2$ and b_9 is a constant independent of h.

Stability of the method

To study the stability of the method, we change $S_\Delta(x)$ to $W_\Delta(x)$ where

$$W_\Delta(x) = W^m_{k-1}(x_k) + W'^m_{k-1}(x_k)(x-x_k) + \int_{x_k}^{x}\int_{x_k}^{t} f(u, W^{m-1}_k(u), W'^{m-1}_k(u), P^{m-1}_k(u))\,du\,dt, \tag{25}$$

$$P^{m-1}_k(u) = \int_b^b K(u, t, W^{m-1}_k(u), W'^{m-1}_k(u))dt,$$

where $W^m_{-1}(x) = c_1$, $W'^m_{-1}(x) = c_2$. In Eq.(25) we use the following m iterations. For $x \in [x_k, x_{k+1}]$, $k = 0,1,...,n-1$ and $j = 1,2,...,m$

$$W^i_k(x) = W^m_{k-1}(x_k) + W'^m_{k-1}(x_k)(x-x_k) + \int_{x_k}^{x}\int_{x_k}^{t} f(u, W^{j-1}_k(u), W'^{j-1}_k(u), P^{j-1}_k(u))\,du\,dt, \tag{26}$$

$$P^{j-1}_k(u) = \int_b^b K(u, t, W^{j-1}_k(u), W'^{j-1}_k(u))dt,$$

$$W^0_k(x) = W^m_{k-1}(x_k) + W'^m_{k-1}(x_k)(x-x_k) + \frac{N_k}{2}(x-x_k)^2, \tag{27}$$

$$N_k = f(u, W^m_{k-1}(u), W'^m_{k-1}(u), \int_a^b K(x_k, t, W^m_{k-1}(t), W'^m_{k-1}(t))dt). \tag{28}$$

Moreover, we use the following notations:

$$e^*(x) = | S_\Delta(x) - W_\Delta(x) |, \quad e_k = | S_\Delta(x_k) - W_\Delta(x_k) |, \tag{29}$$

$$e'^*(x) = | S'_\Delta(x) - W'_\Delta(x) |, \quad e'_k = | S_\Delta(x_k) - W_\Delta(x_k) |. \tag{30}$$

Definition 4

For any $x \in [x_k, x_{k+1}]$, $k = 0,1,..., n-1$ and $j = 1,2,...,m$ we define the operator $T^*_{kj}(x)$ by:

$$T^*_{kj}(x) = | S^{m-j}_k(x) - W^{m-j}_k(x) | + | S'^{m-j}_k(x) - W'^{m-j}_k(x) |,$$

whose norm is defined by:

$$\| T^{*}_{kj} \| = \max_{x \in [x_k, x_{k+1}]} \{ T^{*}_{kj}(x) \}.$$

Lemma 5

For any $x \in [x_k, x_{k+1}]$, $k = 0,1,...,n-1$ and $j = 1,2,...,m$ then:

$$\| T^{*}_{km} \| \le (1 + \frac{3}{2} b_0) e^{*}_k + (2 + \frac{3}{2} b_0) e'^{*}_k, \tag{31}$$

$$\| T^{*}_{k1} \| \le b_1 e^{*}_k + b_2 e'^{*}_k, \tag{32}$$

where b_0, b_1, b_2 are constants defined as in Lemma 3.

Proof

To prove (31), using (4), (5), (8), (27), (29), and (30). To prove (32), using (4), (5), (7), (26), (29), (30), (32), and Lemma 3. The proof is similar to the proof of Lemma 3.

Lemma 6

Let $e^{*}(x)$, $e'^{*}(x)$ be defined as in (29), (30), then there exist constants b_3, b_4, b_5, b_6, independent of h such that the following inequalities hold:

$$e^{*}(x) \le (1 + hb_3) e^{*}_k + hb_4 e'^{*}_k,$$

$$e'^{*}(x) \le hb_5 e^{*}_k + (1 + hb_6) e'^{*}_k,$$

where $b_3 = \frac{b_0 b_1}{2}$, $b_4 = 1 + \frac{b_0 b_1}{2}$, $b_5 = b_0 b_1$, $b_6 = b_0 b_2$.

Proof

Using (3), (4), (5), (25), (29), and (30). The proof is similar to the proof of Lemma 4.

Theorem 2

Let $S_\Delta(x)$ given by (6) is the approximate solution of the problem (1) with the initial conditions $y(x_0) = c_1$ and $y'(x_0) = c_2$ and let $W_\Delta(x)$ given by (25) be the approximate solution for the same problem with initial conditions $y^{*}(x_0) = c^{*}_1$ and $y'^{*}(x_0) = c^{*}_2$, $f \in C[a,b] \times R^4$ then the inequalities:

$$| S^{(q)}_\Delta - W^{(q)}_\Delta | \le b_{10} \| E^{*}_0 \|,$$

hold for $x \in [a,b]$, $q = 0,1,2,$

$$\| E_0^* \| = \max_{x \in [a,b]} \{ | y_0 - y_0^* | + | y'_0 - y'^*_0 | \},$$

where b_{10} is a constant independent of h.

Numerical example

In this section, we consider the following equation:

$$y''(x) = y'(x) - \int_0^1 y'(t)dt + 1.71828, \quad (33)$$

with the initial conditions $y(0)=1$, $y'(0)=1$. The exact solution of this problem is $y(x)=e^x$.

The VIM gives the possibility to write the solution of Eq.(33) with the aid of the correction functional:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)[\, y_{n\tau\tau} - \tilde{y}_{n\tau} + \int_0^1 \tilde{y}_{nt}(t)dt - 1.71828\,]d\tau. \quad (34)$$

It is obvious that the successive approximations y_n, $n \geq 0$ (the subscript n denotes the n-th order approximation), can be established by determining, the general Lagrange multiplier, λ, which can be identified optimally via the variational theory ([2], [9]-[11]). The function $\tilde{y}_n$ is a restricted variation, which means $\delta \tilde{y}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations y_n, $n \geq 1$, of the solution y will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function y_0. The initial values of the solution are usually used for selecting the zeroth approximation y_0. With determined, then several approximations y_n, $n \geq 1$, follow immediately. Consequently, the exact solution may be obtained by using:

$$y(x) = \lim_{n \to \infty} y_n(x). \quad (35)$$

Making the above correction functional stationary, and noticing that $\delta \tilde{y}_n = 0$, we obtain:

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(\tau)[\, y_{n\tau\tau} - \tilde{y}_{n\tau} + \int_0^1 \tilde{y}_{nt}(t)dt - 1.71828\,]d\tau$$

$$= \delta y_n + [\lambda(\tau)\,\delta y_n - \dot{\lambda}(\tau)\delta y_n]_{\tau=x} + \int_0^x \ddot{\lambda}(\tau)[\,\delta y_n]d\tau = 0,$$

where $\delta \tilde{y}_n$ is considered as a restricted variation i.e., $\delta \tilde{y}_n = 0$, yields the following stationary conditions:

$$\ddot{\lambda}(\tau) = 0, \qquad 1 - \lambda(\tau)\,|_{\tau=x} = 0, \qquad 1 - \dot{\lambda}(\tau)\,|_{\tau=x} = 0. \tag{36}$$

The equation (36) is called Lagrange-Euler equation, the Lagrange multiplier, therefore, can be readily identified $\lambda(\tau) = x - \tau$.

Now, the following variational iteration formula can be obtained:

$$y_{n+1}(x) = y_n(x) + \int_0^x (x - \tau)\,[\, y_{n\tau\tau} - y_{n\tau} + \int_0^1 y_{nt}(t)dt - 1.71828]d\tau. \tag{37}$$

We start with an initial approximation and by using the above iteration formula (37), we can obtain directly the components of the solution as follows:

$$y_0(x) = 1 + x,$$

$$y_1(x) = 1 + x + 0.8591x^2,$$

$$y_2(x) = 1 + x + 0.4296x^2 + 0.2864x^3,$$

$$y_3(x) = 1 + x + 0.5011x^2 + 0.1432x^3 + 0.0716x^4,$$

$$y_4(x) = 1 + x + 0.5012x^2 + 0.1671x^3 + 0.03358x^4 + 0.0143x^5,$$

$$y_5(x) = 1 + x + 0.4999x^2 + 0.1671x^3 + 0.0418x^4 + 0.0072x^5 + 0.0023x^6.$$

In the same manner, we can obtain other components of the solution. In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the numerical solutions using $n = 8$ terms approximation.

Table 1, shows the numerical results of the problem (33) using the spline function expansion. In this Table, we compute the first approximate solution (First app. sol.), first absolute error (the difference between the exact and approximate solution before change), the second approximate solution (Scond app. sol.) and the second absolute error (the difference between the first and second solutions), with different iterations number m at some values of $x = 0.1,\ 0.2, 0.3, 0.4, 0.5$.

The above simulation proves that the proposed method is a very useful numerical method to get accurate solutions to second order Fredholm integro-differential equations. Figure 1, presents a comparison between the exact solution, y_{exact}, the solution obtained from the proposed method, y_{spline} and the solution using VIM, y_{VIM} in the interval [0,1]. Figure 1, shows that the proposed method provides excellent approximations to the solution of related equation to second-order Fredholm integro-differential equations. The numerical results showed that the proposed method has very accuracy and reductions of the size of calculations compared with the VIM ([3], [11]).

Concluding remarks and discussion

This paper centralized to present a new method for solving the second order FIDEs. This analysis shows that the proposed technique has much impact on the accuracy and efficiency of the solution on the second order FIDEs. We investigated the error estimation and the stability of the proposed method. The analytical approximation to the solutions is reliable, and confirms the power and ability of the proposed technique as an easy device for computing the solution of such these problems. The presented example shows that the results of the proposed method are in excellent agreement with those of exact solution. Also, a comparison with the approximate method, VIM is given. All computations in this paper are done using Mathematica 6.

Table 1

x	m	First app. sol.	First absolute error	Second app. sol.	Second absolute error
0.1	1	1.10458208	5.9×10^{-4}	1.10459305	5.8×10^{-4}
	2	1.10516199	8.9×10^{-6}	1.10517300	2.1×10^{-6}
	3	1.10518243	1.1×10^{-5}	1.10519343	2.3×10^{-5}
	4	1.10517102	1.0×10^{-7}	1.10518202	1.1×10^{-5}
	5	1.10517063	2.8×10^{-7}	1.10518164	1.1×10^{-5}
0.2	1	1.22061315	7.9×10^{-4}	1.22062508	7.8×10^{-4}
	2	1.22122640	1.7×10^{-4}	1.22123841	1.6×10^{-4}
	3	1.22142326	2.1×10^{-5}	1.22143511	3.6×10^{-5}
	4	1.22124071	4.4×10^{-6}	1.22141914	1.6×10^{-5}
	5	1.22140219	5.6×10^{-7}	1.22141419	1.1×10^{-5}
0.3	1	1.35039036	5.3×10^{-4}	1.35040325	5.4×10^{-4}
	2	1.34493612	4.9×10^{-4}	1.34937428	4.8×10^{-4}
	3	1.34985298		1.34986598	
	4	1.34987245		1.34988545	

	5	1.34985878	5.8×10^{-6} 1.3×10^{-6} 2.8×10^{-8}	1.34987178	7.2×10^{-6} 2.7×10^{-5} 1.3×10^{-5}
0.4	1 2 3 4 5	1.49481911 1.49097150 1.49175776 1.49184910 1.49182610	2.9×10^{-3} 8.5×10^{-4} 6.7×10^{-5} 2.4×10^{-5} 1.4×10^{-6}	1.49832956 1.49098552 1.49177176 1.49186310 1.49184010	3.0×10^{-3} 8.4×10^{-4} 5.3×10^{-5} 3.8×10^{-5} 1.5×10^{-5}
0.5	1 2 3 4 5	1.65503267 1.64753396 1.64856616 1.64875591 1.64872483	6.3×10^{-3} 1.1×10^{-3} 1.5×10^{-4} 3.4×10^{-5} 3.5×10^{-6}	1.65504746 1.64754899 1.64858117 1.64877091 1.64873983	6.3×10^{-3} 1.2×10^{-3} 1.4×10^{-4} 4.9×10^{-5} 1.8×10^{-5}

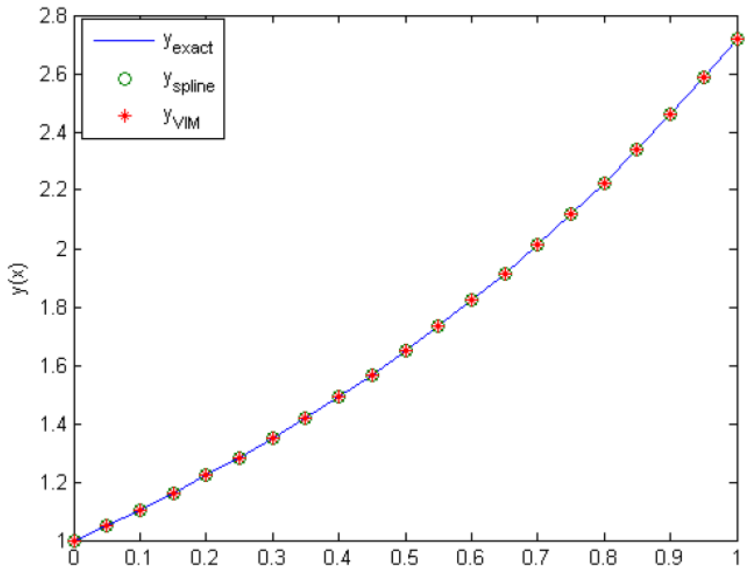

Figure 1: Comparison between the exact solution and solution obtained from the proposed method with the solution using VIM.

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