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Recurrent H-Curvature Tensors in a Para-Sasakian Manifolds

**T.S Chauhan[1], R.C. Dimri[2], V.K. Srivastava[3]
and Indiwar Singh Chauhan[4]**

[1]Deptt. of Mathematics, Bareilly College, Bareilly, U.P., India
2Deptt. of Mathematics, HNB Garhwal University, Srinagar,
Uttaranchal Institute of Technology, Dehradun, Uttarakhand, India
[3]Ganjdundwara P.G. College, Kashiram Nagar, India
[3]Corresponding Author E-mail: vk_srivastava78@yahoo.in

Abstract

This paper presents the theory of recurrent H-curvature tensors in a Para-Sasakian manifolds. Section 1 is devoted to the study of H-curvature tensors. Section 2 deals to the study of recurrent H-curvature tensors. In this section, we have proved that every Para-Sasakian H-projective curvature tensor is a Para-Sasakian manifold with recurrent H-projective curvature tensor.

Keywords: H-Curvature tensor,H-Projective curvature tensor, ricci tensor, bochner curvature tensor & scalar curvature tensor

Introduction

The Riemannian curvature tensor $R^{\lambda}{}_{\alpha\beta\gamma}$ is expressed as

(1.1) $$R^{\lambda}{}_{\alpha\beta\gamma} = \partial_{\beta}\{^{\lambda}_{\alpha\ \gamma}\} - \partial_{\gamma}\{^{\lambda}_{\alpha\ \beta}\} + \{^{\mu}_{\alpha\ \gamma}\}\{^{\lambda}_{\mu\ \beta}\} - \{^{\mu}_{\alpha\ \beta}\}\{^{\lambda}_{\mu\ \gamma}\}$$

Wherein $\partial_{\beta} = \partial/\partial x^{\beta}$.

The Riemannian curvature tensor, the Ricci tensor and the Scalar curvature satisfies the following conditions:

(1.2) $R_{\lambda\alpha\beta\gamma} = g_{\lambda\mu} R^{\mu}{}_{\alpha\beta\gamma}$,

(1.3) $R_{\alpha\beta} = g^{\lambda\gamma} R_{\lambda\alpha\beta\gamma}$,

(1.4) $R_{\alpha\beta} = R^{\lambda}{}_{\alpha\beta\lambda} = R_{\beta\alpha}$,

(1.5) $R^{\lambda}{}_{\alpha\beta\gamma} = -R^{\lambda}{}_{\alpha\gamma\beta}$,

(1.6) $R^{\lambda}{}_{\lambda\alpha\beta} = 0$,

(1.7) $R_{\lambda\alpha\beta\gamma} = -R_{\alpha\lambda\beta\gamma}$,

(1.8) $R_{\lambda\alpha\beta\gamma} = -R_{\lambda\alpha\gamma\beta}$,

(1.9) $R_{\lambda\alpha\beta\gamma} = R_{\beta\gamma\lambda\alpha}$,

(1.10) $R_{\lambda\lambda\beta\gamma} = R_{\lambda\alpha\beta\beta} = 0$,

(1.11) $R_{\alpha\beta} = g_{\lambda\beta} R^{\lambda}{}_{\alpha}$,

(1.12) $R_{\alpha\beta} = \delta^{\lambda}{}_{\beta} R_{\lambda\alpha}$,

(1.13) $R = g^{\alpha\beta} R_{\alpha\beta}$,

(1.14) $R^{\lambda}{}_{\alpha\beta\gamma} + R^{\lambda}{}_{\beta\gamma\alpha} + R^{\lambda}{}_{\gamma\alpha\beta} = 0$

(1.15) $R_{\lambda\alpha\beta\gamma} + R_{\lambda\beta\gamma\alpha} + R_{\lambda\gamma\alpha\beta} = 0$

(1.16) $R^{\lambda}{}_{\alpha\beta\gamma,\varepsilon} + R^{\lambda}{}_{\alpha\gamma\varepsilon,\beta} + R^{\lambda}{}_{\alpha\varepsilon\beta,\gamma} = 0$

and

(1.17) $R_{\lambda\alpha\beta\gamma,\varepsilon} + R_{\lambda\alpha\gamma\varepsilon,\beta} + R_{\lambda\alpha\varepsilon\beta,\gamma} = 0$.

Now, we define a tensor $S_{\alpha\beta}$ by

(1.18) $S_{\alpha\beta} = -F^{\lambda}{}_{\alpha} R_{\lambda\beta}$,

Consequently yields

(1.19) $S_{\alpha\beta} = -S_{\beta\alpha}$,

(1.20) $F^{\lambda}{}_{\alpha} S_{\lambda\beta} = -S_{\alpha\lambda} F^{\lambda}{}_{\beta}$

and

(1.21) $F^{\lambda}{}_{\lambda} = 0$.

Definition 1.1

The Para-Sasakian H-projective curvature tensor is given by

(1.22) $$P^{\lambda}{}_{\alpha\beta\gamma} = R^{\lambda}{}_{\alpha\beta\gamma} + \{1/(n+2)\}(R_{\alpha\gamma}\delta^{\lambda}{}_{\beta}-R_{\alpha\beta}\delta^{\lambda}{}_{\gamma} + S_{\alpha\gamma}F^{\lambda}{}_{\beta}-S_{\alpha\beta}F^{\lambda}{}_{\gamma} + 2S_{\beta\gamma}F^{\lambda}{}_{\alpha}).$$

Contracting equation (1.22) by $g_{\lambda\mu}$, we get

(1.23) $$P_{\mu\alpha\beta\gamma} = R_{\mu\alpha\beta\gamma} + \{1/(n+2)\}(R_{\alpha\gamma}g_{\mu\beta}-R_{\alpha\beta}g_{\mu\gamma} + S_{\alpha\gamma}F_{\mu\beta}-S_{\alpha\beta}F_{\mu\gamma} + 2S_{\beta\gamma}F_{\mu\alpha}).$$

Definition 1.2

The Para-Sasakian H-conformal or Bochner curvature tensor is given by

(1.24) $$B^{\lambda}{}_{\alpha\beta\gamma} = R^{\lambda}{}_{\alpha\beta\gamma} + \{1/(n+4)\}(R_{\alpha\gamma}\delta^{\lambda}{}_{\beta}-R_{\alpha\beta}\delta^{\lambda}{}_{\gamma} + g_{\alpha\gamma}R^{\lambda}{}_{\beta} -g_{\alpha\beta}R^{\lambda}{}_{\gamma} + S_{\alpha\gamma}F^{\lambda}{}_{\beta}-S_{\alpha\beta}F^{\lambda}{}_{\gamma} + F_{\alpha\gamma}S^{\lambda}{}_{\beta}-F_{\alpha\beta}S^{\lambda}{}_{\gamma} + 2S_{\beta\gamma}F^{\lambda}{}_{\alpha} + 2F_{\beta\gamma}S^{\lambda}{}_{\alpha})-\{R/(n+2)(n+4)\}(g_{\alpha\gamma}\delta^{\lambda}{}_{\beta}-g_{\alpha\beta}\delta^{\lambda}{}_{\gamma} + F_{\alpha\gamma}F^{\lambda}{}_{\beta} -F_{\alpha\beta}F^{\lambda}{}_{\gamma} + 2F_{\beta\gamma}F^{\lambda}{}_{\alpha}).$$

Contracting equation (1.24) by $g_{\lambda\mu}$, we get

(1.25) $$B_{\mu\alpha\beta\gamma} = R_{\mu\alpha\beta\gamma}+\{1/(n+4)\}(R_{\alpha\gamma}g_{\mu\beta} -R_{\alpha\beta}g_{\mu\gamma}+g_{\alpha\gamma}R_{\mu\beta} -g_{\alpha\beta}R_{\mu\gamma} + S_{\alpha\gamma}F_{\mu\beta}-S_{\alpha\beta}F_{\mu\gamma} + F_{\alpha\gamma}S_{\mu\beta}-F_{\alpha\beta}S_{\mu\gamma} + 2S_{\beta\gamma}F_{\mu\alpha} + 2F_{\beta\gamma}S_{\mu\alpha})-\{R/(n+2)(n+4)\}(g_{\alpha\gamma}g_{\mu\beta} -g_{\alpha\beta}g_{\mu\gamma} + F_{\alpha\gamma}F_{\mu\beta}-F_{\alpha\beta}F_{\mu\gamma} + 2F_{\beta\gamma}F_{\mu\alpha}).$$

In this regard, we have the following theorems:

Theorem 1.1

The Para-Sasakian H-projective curvature tensor is skew-symmetric in the last two indices i.e.

$$P^{\lambda}{}_{\alpha\beta\gamma} =-P^{\lambda}{}_{\alpha\gamma\beta} .$$

Proof

On interchanging β and γ in equation (1.22), we get

(1.26) $$P^{\lambda}{}_{\alpha\gamma\beta} = R^{\lambda}{}_{\alpha\gamma\beta} + \{1/(n+2)\}(R_{\alpha\beta}\delta^{\lambda}{}_{\gamma}-R_{\alpha\gamma}\delta^{\lambda}{}_{\beta}$$

$+ S_{\alpha\beta}F^{\lambda}{}_{\gamma} - S_{\alpha\gamma}F^{\lambda}{}_{\beta} + 2S_{\gamma\beta}F^{\lambda}{}_{\alpha})$

From equations (1.5), (1.19) and (1.26), we obtain

(1.27) $$P^{\lambda}{}_{\alpha\gamma\beta} = -[R^{\lambda}{}_{\alpha\beta\gamma} + \{1/(n+2)\}(R_{\alpha\gamma}\delta^{\lambda}{}_{\beta} - R_{\alpha\beta}\delta^{\lambda}{}_{\gamma} + S_{\alpha\gamma}F^{\lambda}{}_{\beta} - S_{\alpha\beta}F^{\lambda}{}_{\gamma} + 2S_{\beta\gamma}F^{\lambda}{}_{\alpha})]$$

In view of equations (1.22) and (1.27), we get

(1.28) $$P^{\lambda}{}_{\alpha\beta\gamma} = -P^{\lambda}{}_{\alpha\gamma\beta}$$

Theorem 1.2
Show that $P_{\mu\alpha\beta\gamma} = -P_{\mu\alpha\gamma\beta}$.

Proof
Contracting equation (1.28) by $g_{\lambda\mu}$, we obtain

(1.29) $$P_{\mu\alpha\beta\gamma} = -P_{\mu\alpha\gamma\beta}$$

Theorem 1.3

Prove that $P^{\lambda}{}_{\lambda\alpha\beta} = 0$.

Proof
Contraction of equation (1.22) with regard to the indices λ and α yields

(1.30) $$P^{\lambda}{}_{\lambda\beta\gamma} = R^{\lambda}{}_{\lambda\beta\gamma} + \{1/(n+2)\}(R_{\lambda\gamma}\delta^{\lambda}{}_{\beta} - R_{\lambda\beta}\delta^{\lambda}{}_{\gamma} + S_{\lambda\gamma}F^{\lambda}{}_{\beta} - S_{\lambda\beta}F^{\lambda}{}_{\gamma} + 2S_{\beta\gamma}F^{\lambda}{}_{\lambda})$$

From equations (1.6), (1.12), (1.19) and (1.30), we get

(1.31) $$P^{\lambda}{}_{\lambda\beta\gamma} = \{1/(n+2)\}(-S_{\gamma\lambda}F^{\lambda}{}_{\beta} - S_{\lambda\beta}F^{\lambda}{}_{\gamma} + 2S_{\beta\gamma}F^{\lambda}{}_{\lambda})$$

In view of equations (1.20), (1.21) and (1.31) yields

(1.32) $$P^{\lambda}{}_{\lambda\beta\gamma} = 0$$

Theorem 1.4

Prove that $P^{\lambda}{}_{\alpha\beta\gamma} + P^{\lambda}{}_{\beta\gamma\alpha} + P^{\lambda}{}_{\gamma\alpha\beta} = 0$.

Proof

By virtue of equation (1.22), we get

(1.33) $$P^{\lambda}{}_{\beta\gamma\alpha} = R^{\lambda}{}_{\beta\gamma\alpha} + \{1/(n+2)\}(R_{\beta\alpha}\delta^{\lambda}{}_{\gamma} - R_{\beta\gamma}\delta^{\lambda}{}_{\alpha} + S_{\beta\alpha}F^{\lambda}{}_{\gamma} - S_{\beta\gamma}F^{\lambda}{}_{\alpha} + 2S_{\gamma\alpha}F^{\lambda}{}_{\beta})$$

and

(1.34) $$P^{\lambda}{}_{\gamma\alpha\beta} = R^{\lambda}{}_{\gamma\alpha\beta} + \{1/(n+2)\}(R_{\gamma\beta}\delta^{\lambda}{}_{\alpha} - R_{\gamma\alpha}\delta^{\lambda}{}_{\beta} + S_{\gamma\beta}F^{\lambda}{}_{\alpha} - S_{\gamma\alpha}F^{\lambda}{}_{\beta} + 2S_{\alpha\beta}F^{\lambda}{}_{\gamma})$$

Adding equations (1.22), (1.33) and (1.34), we obtain

(1.35) $$P^{\lambda}{}_{\alpha\beta\gamma} + P^{\lambda}{}_{\beta\gamma\alpha} + P^{\lambda}{}_{\gamma\alpha\beta} = (R^{\lambda}{}_{\alpha\beta\gamma} + R^{\lambda}{}_{\beta\gamma\alpha} + R^{\lambda}{}_{\gamma\alpha\beta}) + \{1/(n+2)\}(R_{\alpha\gamma}\delta^{\lambda}{}_{\beta} - R_{\alpha\beta}\delta^{\lambda}{}_{\gamma} + S_{\alpha\gamma}F^{\lambda}{}_{\beta} - S_{\alpha\beta}F^{\lambda}{}_{\gamma} + 2S_{\beta\gamma}F^{\lambda}{}_{\alpha} + R_{\beta\alpha}\delta^{\lambda}{}_{\gamma} - R_{\beta\gamma}\delta^{\lambda}{}_{\alpha} + S_{\beta\alpha}F^{\lambda}{}_{\gamma} - S_{\beta\gamma}F^{\lambda}{}_{\alpha} + 2S_{\gamma\alpha}F^{\lambda}{}_{\beta} + R_{\gamma\beta}\delta^{\lambda}{}_{\alpha} - R_{\gamma\alpha}\delta^{\lambda}{}_{\beta} + S_{\gamma\beta}F^{\lambda}{}_{\alpha} - S_{\gamma\alpha}F^{\lambda}{}_{\beta} + 2S_{\alpha\beta}F^{\lambda}{}_{\gamma})$$

From equations (1.4), (1.14) and (1.35), we get

(1.36) $$P^{\lambda}{}_{\alpha\beta\gamma} + P^{\lambda}{}_{\beta\gamma\alpha} + P^{\lambda}{}_{\gamma\alpha\beta} = \{1/(n+2)\}(S_{\alpha\gamma}F^{\lambda}{}_{\beta} - S_{\alpha\beta}F^{\lambda}{}_{\gamma} + 2S_{\beta\gamma}F^{\lambda}{}_{\alpha} + S_{\beta\alpha}F^{\lambda}{}_{\gamma} - S_{\beta\gamma}F^{\lambda}{}_{\alpha} + 2S_{\gamma\alpha}F^{\lambda}{}_{\beta} + S_{\gamma\beta}F^{\lambda}{}_{\alpha} - S_{\gamma\alpha}F^{\lambda}{}_{\beta} + 2S_{\alpha\beta}F^{\lambda}{}_{\gamma})$$

In view of equations (1.19) and (1.36) yields

(1.37) $$P^{\lambda}{}_{\alpha\beta\gamma} + P^{\lambda}{}_{\beta\gamma\alpha} + P^{\lambda}{}_{\gamma\alpha\beta} = 0$$

i.e.

(1.38) $$P^{\lambda}_{[\alpha\beta\gamma]} = 0$$

Theorem 1.5
Prove that

$$P_{\mu\alpha\beta\gamma} + P_{\mu\beta\gamma\alpha} + P_{\mu\gamma\alpha\beta} = 0.$$

Proof
Contracting equation (1.37) by $g_{\lambda\mu}$, we obtain

(1.39) $$P_{\mu\alpha\beta\gamma} + P_{\mu\beta\gamma\alpha} + P_{\mu\gamma\alpha\beta} = 0$$

i.e.

(1.40) $$P_{\mu[\alpha\beta\gamma]} = 0$$

Recurrent H-Curvature Tensors

Definition 2.1
The Para-Sasakian manifold is called recurrent if we have

(2.1) $$R^{\lambda}_{\alpha\beta\gamma,\theta} - A_{\theta} R^{\lambda}_{\alpha\beta\gamma} = 0,$$

for some non-zero recurrence vector A_{θ} .

Contracting $R^{\lambda}_{\alpha\beta\gamma,\theta}$ for λ and γ, we get

(2.2) $$R_{\alpha\beta,\theta} - A_{\theta} R_{\alpha\beta} = 0,$$

Contracting equation (2.2) by $g^{\alpha\beta}$, we obtain

(2.3) $$R_{,\theta} - A_{\theta} R = 0.$$

Definition 2.2
The Para-Sasakian manifold satisfies the relation

(2.4) $$P^{\lambda}_{\alpha\beta\gamma,\theta} - A_{\theta} P^{\lambda}_{\alpha\beta\gamma} = 0,$$

for some non-zero recurrence vector A_θ , will be called the Para-Sasakian manifold with recurrent H-projective curvature tensor.

Definition 2.3
The Para-Sasakian manifold satisfies the relation

(2.5) $$B^{\lambda}{}_{\alpha\beta\gamma,\theta} - A_\theta B^{\lambda}{}_{\alpha\beta\gamma} = 0,$$

for some non-zero recurrence vector A_θ , will be called the Para-Sasakian manifold with recurrent H-conformal or Bochner curvature tensor.

In this regard, we have the following theorems:

Theorem 2.1
If the Para-Sasakian manifold is recurrent then every Para-Sasakian H-projective curvature tensor is a Para-Sasakian manifold with recurrent H-projective curvature tensor.

Proof

Differentiating equation (1.22) covariantly with respect to x^θ, we get

(2.6) $$P^{\lambda}{}_{\alpha\beta\gamma,\theta} = R^{\lambda}{}_{\alpha\beta\gamma,\theta} + \{1/(n+2)\}(R_{\alpha\gamma,\theta}\delta^{\lambda}{}_{\beta} - R_{\alpha\beta,\theta}\delta^{\lambda}{}_{\gamma} + S_{\alpha\gamma,\theta}F^{\lambda}{}_{\beta} - S_{\alpha\beta,\theta}F^{\lambda}{}_{\gamma} + 2S_{\beta\gamma,\theta}F^{\lambda}{}_{\alpha})$$

From equations (1.18) and (2.6), we obtain

(2.7) $$P^{\lambda}{}_{\alpha\beta\gamma,\theta} = R^{\lambda}{}_{\alpha\beta\gamma,\theta} + \{1/(n+2)\}(\delta^{\lambda}{}_{\beta} R_{\alpha\gamma,\theta} - \delta^{\lambda}{}_{\gamma} R_{\alpha\beta,\theta} - F^{\lambda}{}_{\beta} F^{\nu}{}_{\alpha} R_{\nu\gamma,\theta} + F^{\lambda}{}_{\gamma} F^{\varepsilon}{}_{\alpha} R_{\varepsilon\beta,\theta} - 2F^{\lambda}{}_{\alpha} F^{\tau}{}_{\beta} R_{\tau\gamma,\theta})$$

By virtue of equations (2.1), (2.2) and (2.7), we get

(2.8) $$P^{\lambda}{}_{\alpha\beta\gamma,\theta} = A_\theta[R^{\lambda}{}_{\alpha\beta\gamma} + \{1/(n+2)\}(\delta^{\lambda}{}_{\beta} R_{\alpha\gamma} - \delta^{\lambda}{}_{\gamma} R_{\alpha\beta} - F^{\lambda}{}_{\beta} F^{\nu}{}_{\alpha} R_{\nu\gamma} + F^{\lambda}{}_{\gamma} F^{\varepsilon}{}_{\alpha} R_{\varepsilon\beta} - 2F^{\lambda}{}_{\alpha} F^{\tau}{}_{\beta} R_{\tau\gamma})]$$

In view of equations (1.18) and (2.8), we obtain

(2.9) $$P^{\lambda}{}_{\alpha\beta\gamma,\theta} = A_\theta[R^{\lambda}{}_{\alpha\beta\gamma} + \{1/(n+2)\}(R_{\alpha\gamma}\delta^{\lambda}{}_{\beta} - R_{\alpha\beta}\delta^{\lambda}{}_{\gamma} + S_{\alpha\gamma}F^{\lambda}{}_{\beta} - S_{\alpha\beta}F^{\lambda}{}_{\gamma} + 2S_{\beta\gamma}F^{\lambda}{}_{\alpha})]$$

From equations (1.22) and (2.9), we get

(2.10) $$P^{\lambda}_{\alpha\beta\gamma,\theta} = A_{\theta} P^{\lambda}_{\alpha\beta\gamma}$$

Hence, we have the desired result.

Theorem 2.2
If the Para-Sasakian manifold is Para-Sasakian manifold with recurrent H-projective curvature tensor then prove that

$$P^{\lambda}_{\lambda\alpha\beta,\theta} = 0.$$

Proof
By virtue of equation (2.4), we get

(2.11) $$P^{\lambda}_{\lambda\alpha\beta,\theta} = A_{\theta} P^{\lambda}_{\lambda\alpha\beta}$$

In view of equations (1.32) and (2.11), we obtai n

(2.12) $$P^{\lambda}_{\lambda\alpha\beta,\theta} = 0$$

Theorem 2.3
If the Para-Sasakian manifold is Para-Sasakian manifold with recurrent H-projective curvature tensor then prove that

$$P^{\lambda}_{\alpha\beta\gamma,\theta} = -P^{\lambda}_{\alpha\gamma\beta,\theta}.$$

Proof
In view of equation (2.4), we obtain

(2.13) $$P^{\lambda}_{\alpha\beta\gamma,\theta} = A_{\theta} P^{\lambda}_{\alpha\beta\gamma}$$

From equations (1.28) and (2.13), we get

(2.14) $$P^{\lambda}_{\alpha\beta\gamma,\theta} = -A_{\theta} P^{\lambda}_{\alpha\gamma\beta}$$

By virtue of equations (2.4) and (2.14), we obtain

(2.15) $$P^{\lambda}_{\alpha\beta\gamma,\theta} = -P^{\lambda}_{\alpha\gamma\beta,\theta}$$

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