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Existence, nonexistence and uniqueness of positive weak solution for a nonlinear system involving weighted p-Laplacian

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Abstract

In this paper, we study the existence, nonexistence and uniqueness of positive weak solution for the nonlinear system

$$\begin{cases} -\Delta_{P,p} u = a(x)[\lambda u^{\alpha} + u^{\beta}] & \text{in } \Omega \\ u > k & \text{in } \Omega \\ u = k & \text{on } \partial\Omega. \end{cases}$$

where $\Delta_{P,p}$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian defined by $\Delta_{P,p} u \equiv div[P(x)|\nabla u|^{p-2}\nabla u]$, λ is a positive parameter, $a(x)$ be a weight function, $0 < \beta \leq \alpha < p-1$, $k \in [0, \infty)$ and $\Omega \subset \Re^N$ is a bounded domain with smooth boundary $\partial\Omega$. For $k \in [0, \infty)$, we establish positive constant $\lambda^*(\Omega)$ such that the above system has a positive weak solution when $\lambda \geq \lambda^*$ while if $k = 1$ and $\lambda + 1 \leq \lambda_1$, the above system has no positive weak solution. Also, we discuss the uniqueness of positive weak solution. We use the method of sub–supersolutions to establish our results.

AMS subject classification: 35J60, 35B40.
Keywords: Positive weak solution, p-Laplacian, nonlinear system, sub-supersolutions.

1. Introduction

In this paper, we are concerned with the existence, nonexistence and uniqueness of positive weak solution for the nonlinear system

$$\begin{cases} -\Delta_{P,p} u = a(x)[\lambda u^{\alpha} + u^{\beta}] & \text{in } \Omega \\ u > k & \text{in } \Omega \\ u = k & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Delta_{P,p}$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian defined by $\Delta_{P,p} u \equiv div[P(x)|\nabla u|^{p-2}\nabla u]$, λ is a positive parameter, $a(x)$ be a weight function and that there exist positive constant a_0 such that $a(x) \geq a_0$, $0 < \beta \leq \alpha < p - 1$, $k \in [0, \infty)$ and $\Omega \subset \Re^N$ is a bounded domain with smooth boundary $\partial\Omega$. For $k \in [0, \infty)$, we establish positive constant $\lambda^*(\Omega)$ such that the above system have a positive weak solution when $\lambda \geq \lambda^*$ while if $k = 1$ and $\lambda + 1 \leq \lambda_1$, the above system has no positive weak solution. Also, we proved the uniqueness of positive weak solution of (1.1). We will use the method of sub–supersolutions to establish our results (see e.g. [4] and [6]).

When $P(x) = a(x) = 1$, problems of the form (1.1) arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mappings (see [17]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids. In the latter case, the quantity p is a characteristic of the medium. The situation $p > 2$ corresponds to dilatant fluids, while the situation $1 < p < 2$ describes pseudo-plastic fluid (see [2]). The case $p = 2$ describes Newtonian fluid.

On the other hand, the existence of weak solutions for nonlinear elliptice systems involving p-Laplacian operators with different weights has been studied using an approximation method (see [8, 9, 15, 16]) and the theory of nonlinear monotone operators method (see [10, 13, 14]).

This paper is organized as follows:

In section 2, we introduce some technical results and notations, which are established in [5]. In section 3, we prove the existence of a positive weak solution for system (1.1) by using the method of sub–supersolutions. Also, we consider the nonexistence result. In section 4, we prove the uniqueness of positive weak solution for system (1.1).

2. Technical results

Now, we introduce some technical results [5] concerning the degenerated homogeneous eigenvalue problem

$$\left.\begin{aligned} -\Delta_{P,p} u = -div[P(x)|\nabla u|^{p-2}\nabla u] = \lambda a(x)|u|^{p-2}u \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.1}$$

where $P(x)$ and $a(x)$ are measurable functions satisfying

$$\frac{\nu(x)}{c_1} \leq P(x) \leq c_1 \nu(x), \tag{2.2}$$

for a.e. $x \in \Omega$ with some constant $c_1 \geq 1$, where $\nu(x)$ is a weight function in Ω satisfying the conditions

$$\nu \in L^1_{Loc}(\Omega),\ \nu^{-\frac{1}{p-1}} \in L^1_{Loc}(\Omega),\ \nu^{-s} \in L^1(\Omega),\ \text{with}\ s \in \left(\frac{N}{p}, \infty\right) \cap \left(\frac{1}{p-1}, \infty\right), \tag{2.3}$$

and

$$0 \leq a(x) \in L^{\frac{k}{k-p}}(\Omega) \quad \text{for a.e. } x \in \Omega, \tag{2.4}$$

with some k satisfies $p < k < p_s^*$ where $p_s^* = \dfrac{Np_s}{N - p_s}$ with $p_s = \dfrac{ps}{s+1} < p < p_s^*$ and *meas* $\{x \in \Omega : a(x) > 0\} > 0$. Examples of functions satisfying (2.3) are mentioned in [5].

Lemma 2.1. There exists the least(i.e. the first or principal) eigenvalue $\lambda_1 > 0$ and precisely one corresponding eigenfunction $\phi_1 \geq 0\ a.e.\ in\ \Omega$ (ϕ_1 not identical to 0) of the eigenvalue problem (2.1). Moreover, it is characterized by

$$\lambda_1 \int_\Omega a(x)\phi_1^p = \int_\Omega P(x)|\nabla\phi_1|^p. \tag{2.5}$$

Lemma 2.2. Let $\phi_1 \in W_0^{1,p}(P, \Omega)$, $\phi_1 \geq 0\ a.e.$ in Ω, be the eigenfunction corresponding to the first eigenvalue $\lambda_1 > 0$ of the eigenvalue problem (2.1). Then $\phi_1 \in L^\infty(\Omega)$.

Now, let us introduce the weighted Sobolev space $W^{1,p}(\nu, \Omega)$ which is the set of all real valued functions u defined in Ω for which (see [5])

$$\|u\|_{1,p,\nu} = \left[\int_\Omega |u|^p + \int_\Omega \nu(x)|\nabla u|^p\right]^{\frac{1}{p}} < \infty. \tag{2.6}$$

Since we are dealing with the Dirichlet problem, we introduce also the space $W_0^{1,p}(\nu, \Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\nu, \Omega)$ with respect to the norm

$$\|u\|_{1,p,\nu} = \left[\int_\Omega \nu(x)|\nabla u|^p\right]^{\frac{1}{p}} < \infty, \tag{2.7}$$

which is equivalent to the norm given by (2.6). Both spaces $W^{1,p}(\nu, \Omega)$ and $W_0^{1,p}(\nu, \Omega)$ are well defined reflexive Banach Spaces.

In this paper, we shall take $c_1 = 1$ in (2.2) i. e. $\nu(x) = P(x)$.

3. Existence and nonexistence results

In this section, we shall prove the existence of positive weak solution for system (1.1) by constructing a positive weak subsolution $\psi \in W_0^{1,p}(P,\Omega)$ and supersolution $z \in W_0^{1,p}(P,\Omega)$ of (1.1) such that $\psi \le z$. That is, ψ satisfies $\psi = 0$ on $\partial\Omega$ and

$$\int_\Omega P(x)|\nabla\psi|^{p-2}\nabla\psi\nabla\zeta\,dx \le \int_\Omega a(x)[\lambda\psi^\alpha + \psi^\beta]\zeta\,dx, \tag{3.1}$$

and z satisfies $z = 0$ on $\partial\Omega$ and

$$\int_\Omega P(x)|\nabla z|^{p-2}\nabla z\nabla\zeta\,dx \ge \int_\Omega a(x)[\lambda z^\alpha + z^\beta]\zeta\,dx, \tag{3.2}$$

for all test function $\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \ge 0$.

Then the following result holds:

Lemma 3.1. (see [3, 11]) Suppose there exist a weak subsolution ψ and a weak supersolution z of (1.1) such that $\psi \le z$; then there exists a weak solution u of (1.1) such that $\psi \le u \le z$.

Our main results of this paper are the following theorems.

Theorem 3.2. There exists positive constant $\lambda^* = \lambda^*(\Omega)$ such that system (1.1) has a positive weak solution u for $\lambda \ge \lambda^*$.

Theorem 3.3. When $k = 1$ and $\lambda + 1 \le \lambda_1$, then system (1.1) has no positive weak solution.

Proof of Theorem 3.2. Let λ_1 be the first eigenvalue of the eigenvalue problem (2.1) and ϕ_1 the corresponding positive eigenfunction satisfying $\phi_1 > 0$ in Ω and $|\nabla\phi_1| > 0$ on $\partial\Omega$ with $\|\phi_1\|_\infty = 1$. Then we have

$$\begin{cases} -\Delta_{P,p}\phi_1 = \lambda_1 a(x)|\phi_1|^{p-1} & \text{in } \Omega \\ \phi_1 > k & \text{in } \Omega \\ \phi_1 = k & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Also, let $m, \delta, \sigma > 0$ be such that $P(x)|\nabla\phi_1|^p - \lambda_1 a(x)\phi_1^p \ge m$ on $\overline{\Omega}_\delta = \{x \in \Omega : d(x,\partial\Omega) \le \delta\}$ and $\phi_1 \ge \sigma > k$ in $\Omega - \overline{\Omega}_\delta$.

We shall verify that $\psi = \left(\frac{p-1}{p}\right)\phi_1^{\frac{p}{p-1}}$ is a weak subsolution of (1.1). Let $\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \ge 0$.

A calculation shows that

$$\begin{aligned}\int_{\Omega} P(x)|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\zeta dx &= \int_{\Omega} P(x)\phi_1|\nabla\phi_1|^{p-2}\nabla\phi_1\cdot\nabla\zeta dx\\ &= \int_{\Omega}(P(x)|\nabla\phi_1|^{p-2}\nabla\phi_1\nabla(\phi_1\zeta) - P(x)|\nabla\phi_1|^{p}\zeta)dx\\ &= \int_{\Omega}(\lambda_1 a(x)\phi_1^p - P(x)|\nabla\phi_1|^p)\zeta dx.\end{aligned}$$

Now , in $\overline{\Omega}_\delta$ we have $\lambda_1 a(x)\phi_1^p - P(x)|\nabla\phi_1|^p \le -m$. Then we have

$$\int_{\overline{\Omega}_\delta} P(x)|\nabla\psi|^{p-2}\nabla\psi\nabla\zeta dx \le 0 \le \int_{\overline{\Omega}_\delta} a(x)[\lambda\psi^\alpha + \psi^\beta]\zeta dx.$$

Next, in $\Omega - \overline{\Omega}_\delta$ we have $\lambda_1 a(x)\phi_1^p - P(x)|\nabla\phi_1|^p \le \lambda_1$ and $\phi_1 \ge \sigma$. Now if we take

$$\lambda \ge \lambda^* = \frac{p^\alpha\lambda_1}{a_0[(p-1)\sigma^{\frac{p}{p-1}}]^\alpha}, \tag{3.4}$$

then we have

$$\begin{aligned}\int_{\Omega-\overline{\Omega}_\delta} P(x)|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\zeta dx &= \int_{\Omega-\overline{\Omega}_\delta}(\lambda_1 a(x)\phi_1^p - P(x)|\nabla\phi_1|^p)\zeta dx\\ &\le \int_{\Omega-\overline{\Omega}_\delta}\lambda a_0\left[\frac{(p-1)\sigma^{\frac{p}{p-1}}}{p}\right]^\alpha \zeta dx\\ &\le \int_{\Omega-\overline{\Omega}_\delta}\lambda a(x)\psi^\alpha dx\\ &\le \int_{\Omega-\overline{\Omega}_\delta} a(x)[\lambda\psi^\alpha + \psi^\beta]dx.\end{aligned}$$

So, equation (3.1) is satisfy and ψ is a weak subsolution of (1.1).

Next, we construct a weak supersolution z of system (1.1). Let $e_p = e_p(x)$ be the positive weak solution of (see [16])

$$\left.\begin{aligned} -\Delta_{P,p}e_p = 1 \quad &\text{in } \Omega,\\ e_p = 0 \quad &\text{on } \partial\Omega,\end{aligned}\right\} \tag{3.5}$$

We denote $z(x) = Ae_p$ where the constant $A > 0$ is sufficiently large and to be chosen later. We shall verify that z is the weak supersolution of (1.1). To do this, let

$\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \geq 0$. Then, using (3.5), we have

$$\begin{aligned}\int_\Omega P(x)|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta\,dx &= A^{p-1}\int_\Omega P(x)|\nabla e_p|^{p-2}\nabla e_p\cdot\nabla\zeta\,dx\\ &= A^{p-1}\int_\Omega \zeta\,dx.\end{aligned}$$

Since $0<\beta\leq\alpha<p-1$, then it is easy to prove that there exists positive large constant A such that

$$A^{p-1-\alpha}=\mu(\lambda e_p^\alpha+A^{\beta-\alpha}e_p^\beta),$$

where $\mu=\|a(x)\|_\infty$. Hence, we have

$$\begin{aligned}\int_\Omega P(x)|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta\,dx &= \int_\Omega \mu(\lambda A^\alpha e_p^\alpha+A^\beta e_p^\beta)\zeta\,dx\\ &\geq \int_\Omega^\alpha a(x)[\lambda z^\alpha+z^\beta]\zeta\,dx.\end{aligned}$$

So, equation (3.2) is satisfy and z is the weak supersolution of (1.1). Thus, there exists a weak solution u of (1.1) with $\psi\leq u\leq z$. This completes the proof of Theorem 3.2.■

Proof of Theorem 3.3. Suppose $u(x)\in W_0^{1,p}(P,\Omega)$ be a positive weak solution of (1.1). We prove Theorem 4 by arriving at a contradiction.

Multiplying (1.1) by u, we have

$$\begin{aligned}\int_\Omega P(x)|\nabla u|^p dx &= \int_\Omega a(x)(\lambda u^{\alpha+1}+u^{\beta+1})dx\\ &< \int_\Omega a(x)(\lambda+1)u^p dx. \qquad (3.6)\end{aligned}$$

Also, we have

$$\lambda_1\int_\Omega a(x)u^p\leq\int_\Omega P(x)|\nabla u|^p. \qquad (3.7)$$

Combining (3.1) and (3.7), we obtain

$$(\lambda_1-(\lambda+1))\int_\Omega a(x)u^p<0,$$

which is a contradiction if $\lambda+1\leq\lambda_1$. Thus system (1.1) has no positive weak solution for $k=1$ and $\lambda+1\leq\lambda_1$, and we finish the proof of Theorem 3.3. ■

Remark 3.4. When $k=1$, the condition (3.4) reduces to $\lambda\geq\lambda^*=\dfrac{p^\alpha\lambda_1}{a_0[(p-1)\sigma]^\alpha}$.

4. Uniqueness of the weak solution

In this section, using a method of [1, 12] we prove the uniqueness of positive weak solution of (1.1)

Theorem 4.1. Let u be the positive weak solution of (1.1). Then it is the unique positive weak solution of (1.1).

Proof. The proof is partly adapted from [4, 5]. Let us assume that $u, v \in W_0^{1,p}(P,\Omega)$ are two positive weak solutions of (1.1). Then it follows from Lemma 2 that $u, v \in L^\infty(\Omega)$ and from (1.1) we have

$$\int_\Omega P(x)|\nabla u|^{p-2}\nabla u \nabla \zeta dx = \int_\Omega a(x)(\lambda u^\alpha + u^\beta)\zeta dx, \tag{4.1}$$

for any $\zeta \in W_0^{1,p}(P,\Omega)$, and

$$\int_\Omega P(x)|\nabla v|^{p-2}\nabla v \nabla \eta dx = \int_\Omega a(x)(\lambda v^\alpha + v^\beta)\eta dx, \tag{4.2}$$

for any $\eta \in W_0^{1,p}(P,\Omega)$. For $\varepsilon > 0$, set $u_\epsilon = u + \varepsilon$, $v_\varepsilon = v + \varepsilon$ and

$$\zeta = \frac{u_\epsilon^p - v_\varepsilon^p}{u_\epsilon^{p-1}}, \quad \eta = \frac{v_\varepsilon^p - u_\epsilon^p}{v_\varepsilon^{p-1}}.$$

Since $\dfrac{u_\epsilon}{v_\varepsilon}, \dfrac{v_\varepsilon}{u_\epsilon} \in L^\infty(\Omega)$ by Lemma 2 and

$$\nabla\zeta = \left[1 + (p-1)\left(\frac{v_\varepsilon}{u_\epsilon}\right)^p\right]\nabla u - p\left(\frac{v_\varepsilon}{u_\epsilon}\right)^{p-1}\nabla v,$$

$$\nabla\eta = \left[1 + (p-1)\left(\frac{u_\varepsilon}{v_\epsilon}\right)^p\right]\nabla v - p\left(\frac{u_\varepsilon}{v_\epsilon}\right)^{p-1}\nabla u,$$

we have $\nabla\zeta, \nabla\eta \in W_0^{1,p}(P,\Omega)$. Adding (4.1) and (4.2) (with ζ and η chosen above) and using the fact that

$$\nabla u_\varepsilon = \nabla u = u_\varepsilon \left|\nabla \log u_\varepsilon\right|, \ \nabla v_\varepsilon = \nabla v = v_\varepsilon \left|\nabla \log v_\varepsilon\right|,$$

we obtain (similarly as in [5] P. 118-120) for $p \geq 2$,

$$\lambda \int_\Omega a(x) \left[\frac{u^\alpha}{u^{p-1}} \left(\frac{u}{u_\varepsilon} \right)^{p-1} - \frac{v^\alpha}{v^{p-1}} \left(\frac{v}{v_\varepsilon} \right)^{p-1} \right] (u_\varepsilon^p - v_\varepsilon^p) dx$$

$$+ \int_\Omega a(x) \left[\frac{u^\beta}{u^{p-1}} \left(\frac{u}{u_\varepsilon} \right)^{p-1} - \frac{v^\beta}{v^{p-1}} \left(\frac{v}{v_\varepsilon} \right)^{p-1} \right] (u_\varepsilon^p - v_\varepsilon^p) dx$$

$$\geq \frac{1}{2^{p-1}-1} \int_\Omega P(x) \left(\frac{1}{v_\varepsilon^p} + \frac{1}{u_\varepsilon^p} \right) |v_\varepsilon \nabla u - u_\varepsilon \nabla v|^p \, dx \geq 0, \tag{4.3}$$

and for $1 < p < 2$,

$$\lambda \int_\Omega a(x) \left[\frac{u^\alpha}{u^{p-1}} \left(\frac{u}{u_\varepsilon} \right)^{p-1} - \frac{v^\alpha}{v^{p-1}} \left(\frac{v}{v_\varepsilon} \right)^{p-1} \right] (u_\varepsilon^p - v_\varepsilon^p) dx$$

$$\geq \frac{3p(p-1)}{16} \int_\Omega P(x) \left(\frac{1}{u_\varepsilon^p} + \frac{1}{v_\varepsilon^p} \right) \frac{|v_\varepsilon \nabla u - u_\varepsilon \nabla v|^2}{|v_\varepsilon |\nabla u| + u_\varepsilon |\nabla v||^{2-p}} dx \geq 0. \tag{4.4}$$

For $\varepsilon \to 0^+$, we have $\dfrac{u}{u_\varepsilon} \to 1$, $\dfrac{v}{v_\varepsilon} \to 1$ $a.e.$ in Ω. Since $0 < \beta \leq \alpha < p-1$, we have for any p, $1 < p < \infty$,

$$\lambda \int_\Omega a(x) \left[\frac{u^\alpha}{u^{p-1}} \left(\frac{u}{u_\varepsilon} \right)^{p-1} - \frac{v^\alpha}{v^{p-1}} \left(\frac{v}{v_\varepsilon} \right)^{p-1} \right] (u_\varepsilon^p - v_\varepsilon^p) dx \leq 0,$$

$$\int_\Omega a(x) \left[\frac{u^\beta}{u^{p-1}} \left(\frac{u}{u_\varepsilon} \right)^{p-1} - \frac{v^\beta}{v^{p-1}} \left(\frac{v}{v_\varepsilon} \right)^{p-1} \right] (u_\varepsilon^p - v_\varepsilon^p) dx \leq 0.$$

Hence it follows from (4.3), (4.1) and from the Fatou lemma that

$$|v \nabla u - u \nabla v| = 0 \quad a.e. \text{ in } \Omega,$$

for any $1 < p < \infty$. Hence there exists a constant $l > 0$ such that $u = lv$ $a.e.$ in Ω. By continuity $u = lv$ at every point in Ω.

Then (4.1) and (4.2) imply that

$$\int_\Omega a(x) \frac{\lambda (lv)^\alpha + (lv)^\beta}{(lv)^{p-1}} v^{p-1} \eta dx = \int_\Omega a(x) \frac{\lambda v^\alpha + v^\beta}{v^{p-1}} v^{p-1} \eta dx,$$

for any $\eta \in W_0^{1,p}(P, \Omega)$ which implies $l = 1$ due to $0 < \beta \leq \alpha < p-1$. This completes the proof of Theorem 4.1. ■

Remark 4.2. When $\beta = 0$ in system (1.1), we have some results presented in [7].

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References

[1] Anane, A., 1987, "Simplicité et isolation de la première valeur propre du p-laplacien avec poids", C. R. Acad. Sei. Paris Sér. I Math., 305, pp. 725–728.

[2] Atkinson, C. and Ali, K., 1992, "Some boundary value problems for the Bingham model", J. Non-Newton. Fluid Mech., 41, pp. 339–363.

[3] Canada, A., Dravek, P. and Gamez, J., 1997, "Existence of positive solutions for some problems with nonlinear diffusion", Trans. Amer. Math. Soc., 349(10), pp. 4231–4249.

[4] Drabek, P. and Hernandez, J., 2001, "Existence and uniqueness of positive solutions for some quasilinear elliptic problem" Nonlinear Anal., 44, pp. 189–204.

[5] Drabek, P., Kufner, A., and Nicolosi, F., 1997, Quasilinear elliptic equation with degenerations and singularities, Walter de Gruyter, Bertin, New York.

[6] Drabek, P., Kerjci, P. and Takac, P., 1999, Nonlinear Differential Equations, Chapman and Hall/CRC.

[7] Khafagy, S., "Existence and uniqueness of positive weak solution for a degenerate nonlinear elliptic system involving p-Laplacian with weight" submitted to New Zealand Journal of Mathematics.

[8] Khafagy, S., 2009, "Maximum Principle and Existence of Positive Solutions for Nonlinear System Involving Different Degenerated p-Laplacian Operators on R^N", New Zealand Journal of Mathematics, 39, pp. 151–163.

[9] Khafagy, S. and Serag, H., 2007, "Maximum Principle and Existence of Positive Solutions for Nonlinear Systems Involving Degenerated p-Laplacian Operators", Electron. J. Diff. Eqns., 2007(66), pp. 1–14.

[10] Khafagy, S. and Serag, H., 2009, "Existence of Weak Solutions for $n \times n$ Nonlinear Systems Involving Different p-Laplacian Operators", Electron. J. Diff. Eqns., 2009(81), pp. 1–14.

[11] Lee, E., Shivaji, R. and Ye, J., 2009, "Positive solutions for elliptic equations involving nonlinearities with falling zeroes" Applied Mathematics Letters, 22, pp, 846–851.

[12] Lindqvist, P., 1990, "On the equation $div(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$" Proc. Amer. Math. Soc., 109, pp. 157–164.

[13] Serag, H. and Khafagy, S., 2008, "Existence of Weak Solutions for $n \times n$ Nonlinear Systems Involving Different Degenerated p-Laplacian Operators", New Zealand Journal of Mathematics, 38, pp. 75–86.

[14] Serag, H. and Khafagy, S., 2009, "Existence of Weak Solutions for Nonlinear Systems Involving Degenerated p-Laplacian Operators", Sarajevo Journal of Mathematics, 5(1), pp. 41–54.
[15] Serag, H. and Khafagy, S., 2009, "On a nonhomogeneous elliptic systems $n \times n$ Involving p-Laplacian with different weights", Journal of Advanced Research in Differential Equations, 1(1), pp. 47–62.
[16] Serag, H. and Khafagy, S., 2010, "On Maximum Principle and Existence of Positive Weak Solutions for $n \times n$ Nonlinear Systems Involving Degenerated p-Laplacian Operator", Turk J M, 34, pp. 59–71.
[17] Tolksdorf, P., 1984, "Regularity for a more general class of quasilinear elliptic equations", J. Diff. Eqns., 51, pp. 126–150.