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A Fibering Map Approach to Qausilinear Elliptic Boundary Value Problem

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Abstract

Using fibering method, we prove the existence of multiple positive solutions of qausilinear problem

$$\begin{cases} -\Delta_p u(x) = \lambda a(x)|u|^{\alpha-1}u + b(x)|u|^{\gamma-1}u & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases}$$

where λ and α are real parameters, Ω is an open bounded domain in $\mathbb{R}^N$, $N \geq 3$, with the smooth boundary $\partial\Omega$, $a, b : \bar{\Omega} \to \mathbb{R}$ are smooth sign changing functions.

The existence results are obtained by the variational method.

AMS subject classification:
Keywords: Variational method, Nehari manifold, Fibering maps, minimizing sequence.

1. Introduction

In this paper we study the existence of positive solutions of the Dirichlet boundary value problem:

$$\begin{cases} -\Delta_p u(x) = \lambda a(x)|u|^{\alpha-1}u + b(x)|u|^{\gamma-1}u & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded region with smooth boundary in $\mathbb{R}^n$, where $\Delta_p(x) = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian $\lambda > 0$ is a real parameter, $1 < \alpha + 1 < 2 < p < \gamma + 1 < p^*$, where

$$p^* = \frac{np}{n-p} \quad for \quad p < n \quad and \quad p^* = \infty \quad for \quad p \geq n$$

and $a, b : \bar{\Omega} \to \mathbb{R}$ are smooth sign changing functions.

Equation (1.1) had been studied by Figueiredo et al. in the case $p = 2$ by using the Mountain Pass lemma [2] and by Il'yasova et al. and Afrouzi et al. by using the Negari manifold [5],[6] and [7]. Furthermore this problem in the case $p = 2$ has been studied by Brown and Wu [8].

In [4] and [3] the results are obtained by using fibering maps (i.e maps of the from $t \to J_\lambda(tu)$) which are closely related to the Nehari manifold. In this paper we show how a fairly complete knowledge of all possible froms of the fibering maps provides a very simple and comparatively elementary means of establishing results similar to those proved in [5] and [7] on the existence of multiple solutions of (1.1). The plan of the paper is as follows:

In section 2 we recall the properties which we shall require of fibering maps and of the Nehari manifold. In section 3 we give a fairly complete description of the fibering maps associated with (1.1) and in section 4 we use this information to give a very simple variational proof of the existence of at least two positive solutions of (1.1) for sufficiently small λ.

2. Notation and Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$. We will work in the Sobolev space $W := W_0^{1,p}(\Omega)$ equipped with the norm

$$\|u\|_W = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

First we give the definition of the weak solution of (1.1).

Definition 2.1. We say that $u \in W$ is a positive weak solution to (1.1) if for any $v \in W$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int_\Omega a(x)|u|^\alpha v dx + \int_\Omega b(x)|u|^\gamma v dx.$$

It is clear that problem (1.1) has a variational structure. Let $J_\lambda : W \to \mathbb{R}$ be the corresponding Euler functional of problem (1.1) which is defined by:

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{\alpha+1}\int_\Omega a(x)|u|^{\alpha+1} dx - \frac{1}{\gamma+1}\int_\Omega b(x)|u|^{\gamma+1} dx. \tag{2.1}$$

It is well known that the weak solutions of Eq. (1.1) are the critical points of the Euler functional J_λ.

When J_λ is bounded below on W; J_λ has a minimizer on W which is a critical point of J_λ. In many problems such as (1.1) J_λ is not bounded below on W but is bounded below on an appropriate subset of W and a minimizer on this set (if it exists) may give rise to a solution of the corresponding differential equation.

A good candidate for an appropriate subset of W is the so-called Nehari manifold

$$M_\lambda(\Omega) = \{u \in W : \langle J_\lambda'(u), u\rangle = 0\},$$

where $\langle , \rangle$ denotes the usual duality between W and W^*. It is clear that all critical points of J_λ must lie on $M_\lambda(\Omega)$ and, as we will see below, local minimizers on $M_\lambda(\Omega)$ are usually critical points of J_λ.

It is easy to see that $u \in M_\lambda(\Omega)$ if and only if

$$\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega a(x)|u|^{\alpha+1} dx - \int_\Omega b(x)|u|^{\gamma+1} dx = 0.$$

Hence if $u \in M_\lambda(\Omega)$, then

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{\alpha+1}\right)\int_\Omega |\nabla u|^p dx + \left(\frac{1}{\alpha+1} - \frac{1}{\gamma+1}\right)\int_\Omega b(x)|u|^{\gamma+1} dx \\ &= \left(\frac{1}{p} - \frac{1}{\gamma+1}\right)\int_\Omega |\nabla u|^p dx - \lambda\left(\frac{1}{\alpha+1} - \frac{1}{\gamma+1}\right)\int_\Omega a(x)|u|^{\alpha+1} dx \end{aligned} \tag{2.2}$$

The Nehari manifold is closely linked to the behaviour of the functions of the form $\phi_u : t \longmapsto J_\lambda(tu)$ $(t > 0)$. Such maps are known as fibering maps and were introduced by Drabek and Pohozaev in [4] and are also discussed in Brown and Zhang [3].

It is clear that if u is a local minimizer of J_λ, then ϕ_u has a local minimum at $t = 1$.

Theorem 2.2. **[3]** Let $u \in W - \{0\}$ and $t > 0$. Then $tu \in M_\lambda(\Omega)$ if and only if $\phi_u'(t) = 0$.

It is easy to see that $u \in M_\lambda(\Omega)$ if and only if $\phi_u'(1) = 0$.

If $u \in W$, we have

$$\begin{aligned} \phi_u(t) = \frac{1}{p}t^p \int_\Omega |\nabla u|^p dx - \lambda\frac{t^{\alpha+1}}{\alpha+1}\int_\Omega a(x)|u|^{\alpha+1} dx \\ - \frac{t^{\gamma+1}}{\gamma+1}\int_\Omega b(x)|u|^{\gamma+1} dx, \end{aligned} \tag{2.3}$$

$$\phi_u'(t) = t^{p-1}\int_\Omega |\nabla u|^p dx - \lambda t^\alpha \int_\Omega a(x)|u|^{\alpha+1} dx - t^\gamma \int_\Omega b(x)|u|^{\gamma+1} dx, \tag{2.4}$$

$$\phi_u''(t) = (p-1)t^{p-2}\int_\Omega |\nabla u|^p dx - \lambda\alpha t^{\alpha-1}\int_\Omega a(x)|u|^{\alpha+1}dx$$
$$- \gamma t^{\gamma-1}\int_\Omega b(x)|u|^{\gamma+1}dx. \tag{2.5}$$

Thus points in $M_\lambda(\Omega)$ correspond to stationary points of fibering maps ϕ_u and so it is natural to divide $M_\lambda(\Omega)$ three subsets $M_\lambda^+(\Omega)$, $M_\lambda^-(\Omega)$ and $M_\lambda^0(\Omega)$ corresponding to local minima, local maxima and points of inflexion of fibering maps.

Hence we define:

$$\begin{aligned} M_\lambda^+(\Omega) &= \{u \in M_\lambda(\Omega) : \phi_u''(1) > 0\},\\ M_\lambda^-(\Omega) &= \{u \in M_\lambda(\Omega) : \phi_u''(1) < 0\},\\ M_\lambda^0(\Omega) &= \{u \in M_\lambda(\Omega) : \phi_u''(1) = 0\}. \end{aligned}$$

Note that if $u \in M_\lambda(\Omega)$, i.e., $\phi_u'(1) = 0$, then

$$\begin{aligned} \phi_u''(1) &= (p-\alpha-1)\int_\Omega |\nabla u|^p dx - (\gamma-\alpha)\int_\Omega b(x)|u|^{\gamma+1}dx\\ &= (p-\gamma-1)\int_\Omega |\nabla u|^p dx + \lambda(\gamma-\alpha)\int_\Omega a(x)|u|^{\alpha+1}dx \end{aligned} \tag{2.6}$$

Also as, proved in Binding, Drabek and Huang [1] or in Brown and Zhang [3], we have the following Lemma.

Lemma 2.3. Suppose that u_0 is a local maximum or minimum for J_λ on $M_\lambda(\Omega)$.
Then, if $u_0 \notin M_\lambda^0(\Omega)$, u_0 is a critical point of J_λ.

Lemma 2.4. J_λ is coercive and bounded below on $M_\lambda(\Omega)$.

Proof. It follows from (2.2) and the Sobolev embedding theorems that there exist positive constants c_1, c_2 and c_3 such that

$$J_\lambda(u) \geq c_1\|u\|_W^p - c_2\int_\Omega |u|^{\alpha+1}dx \geq c_1\|u\|_W^p - c_3\|u\|_W^{\alpha+1}$$

and so J_λ is coercive and bounded below on $M_\lambda(\Omega)$. ■

Define

$$m_u(t) = t^{p-\alpha-1}\int_\Omega |\nabla u|^p dx - t^{\gamma-\alpha}\int_\Omega b(x)|u|^{\gamma+1}dx$$

Then for $t > 0$, $tu \in M_\lambda(\Omega)$ if and only if t is a solution of

$$m_u(t) = \lambda\int_\Omega a(x)|u|^{\alpha+1}dx. \tag{2.7}$$

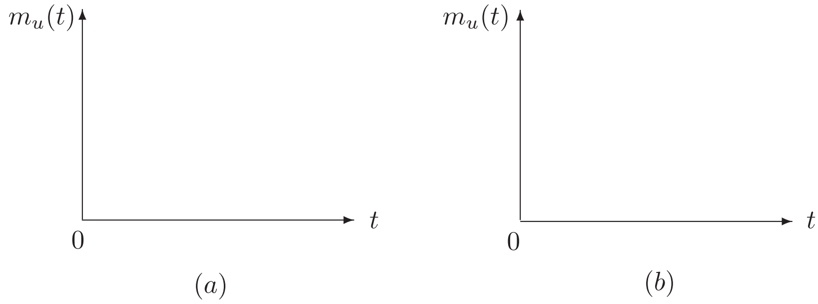

Figure 1: Possible forms of m_u

Moreover,

$$m_u'(t) = (p-\alpha-1)t^{p-\alpha-2}\int_\Omega |\nabla u|^p dx - (\gamma-\alpha)t^{\gamma-\alpha-1}\int_\Omega b(x)|u|^{\gamma+1}dx \tag{2.8}$$

Theorem 2.5.

(i) If $\int_\Omega b(x)|u|^{\gamma+1}dx \le 0$, m_u is a strictly increasing function for $t \ge 0$.

(ii) If $\int_\Omega b(x)|u|^{\gamma+1}dx > 0$, $m_u(t) > 0$ for t small and positive but $m_u(t) \to -\infty$ as $t \to \infty$, also $m_u(t)$ has a unique (maximum) stationary point. (see Fig.1)

Lemma 2.6.

(i) Suppose $tu \in M_\lambda(\Omega)$. Then $\phi_u''(t) = t^\alpha m_u'(t)$.

(ii) If $m_u'(t) > 0(<0)$, then $tu \in M_\lambda^+(\Omega)(M_\lambda^-(\Omega))$.

We shall now describe the nature of the fibering maps for all possible signs of $\int_\Omega a(x)|u|^{\alpha+1}dx$ and $\int_\Omega b(x)|u|^{\gamma+1}dx$. We have the following results.

(i) If $\int_\Omega a(x)|u|^{\alpha+1}dx \le 0$ and $\int_\Omega b(x)|u|^{\gamma+1}dx \le 0$, ϕ_u is an increasing function of t. And so no multiple of u lies in $M_\lambda(\Omega)$. (see Fig 2(a)).

(ii) If $\int_\Omega a(x)|u|^{\alpha+1}dx > 0$ and $\int_\Omega b(x)|u|^{\gamma+1}dx \le 0$, $\phi_u(t) < 0$ for t small and positive but $\phi_u(t) \to +\infty$ as $t \to \infty$, also there is exactly one solution of (2.7). Thus there is a unique value $t(u) > 0$ such that $t(u)u \in M_\lambda^+(\Omega)$. Hence ϕ_u has a unique critical point at $t = t(u)$ which is a local minimum. (see Fig.2(b)).

(iii) If $\int_\Omega a(x)|u|^{\alpha+1}dx \le 0$ and $\int_\Omega b(x)|u|^{\gamma+1}dx > 0$, $\phi_u(t) > 0$ for t small and positive but $\phi_u(t) \to -\infty$ as $t \to \infty$, also there is exactly one solution of (2.7). Thus there is a unique value $t(u) > 0$ such that $t(u)u \in M_\lambda^-(\Omega)$. Hence ϕ_u has a unique critical point at $t = t(u)$ which is a local maximum. (see Fig.2(c)).

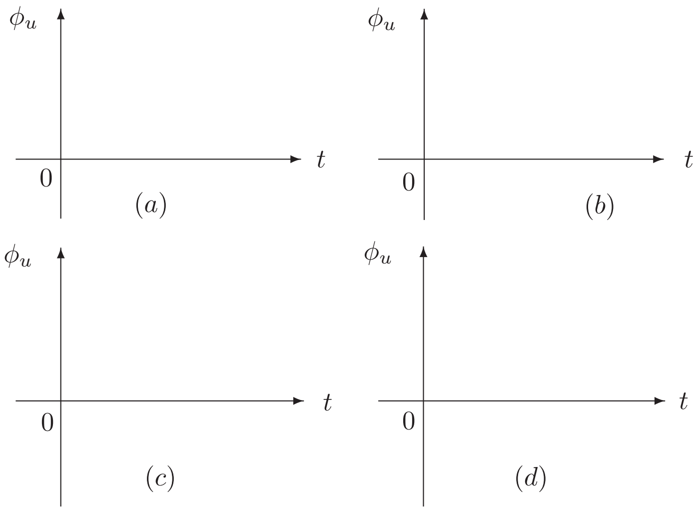

Figure 2: Possible forms of fibering maps.

(iv) If $\int_\Omega a(x)|u|^{\alpha+1}dx > 0$ and $\int_\Omega b(x)|u|^{\gamma+1}dx > 0$,

a) If $\lambda > 0$ is sufficiently large, (2.7) has no solution and so ϕ_u has no critical points, in case ϕ_u is a decreasing function. Hence no multiple of u lies in $M_\lambda(\Omega)$.

b) If $\lambda > 0$ is sufficiently small, there are exactly two solutions $t_1(u) < t_2(u)$ of (2.7) with $m'_u(t_1(u)) > 0$ and $m'_u(t_2(u)) < 0$. Thus there are exactly two multiples of $u \in M_\lambda(\Omega)$, namely $t_1(u)u \in M^+_\lambda(\Omega)$ and $t_2(u)u \in M^-_\lambda(\Omega)$. It follows that ϕ_u has exactly two points - a local minimum at $t = t_1(u)$ and a local maximum at $t = t_2(u)$; moreover ϕ_u is decreasing in $(0, t_1)$, increasing in (t_1, t_2) and decreasing in (t_2, ∞). (see Fig 2(d)).

The following result ensures that when λ is sufficiently small the graph of ϕ_u must be as shown in Figure 2$(a - d)$ for all non-zero u.

Lemma 2.7. There exists $\lambda_1 > 0$ such that, when $\lambda < \lambda_1$, ϕ_u takes on positive values for all non-zero $u \in W$.

Proof. If $\int_\Omega b(x)|u|^{\gamma+1}dx \le 0$, then $\phi_u(t) > 0$ for t sufficiently large. Suppose $u \in W$ and $\int_\Omega b(x)|u|^{\gamma+1}dx > 0$. Let

$$h_u(t) = \frac{1}{p}t^p \int_\Omega |\nabla u|^p dx - \frac{t^{\gamma+1}}{\gamma+1}\int_\Omega b(x)|u|^{\gamma+1}dx.$$

Hence

$$h_u'(t) = t^{p-1}\int_\Omega |\nabla u|^p dx - t^\gamma \int_\Omega b(x)|u|^{\gamma+1}dx,$$

and so if we have $h_u'(t) = 0$ then $t^{p-1}\int_\Omega |\nabla u|^p dx - t^\gamma \int_\Omega b(x)|u|^{\gamma+1}dx = 0$ and so

$$t^{\gamma-p+1} = \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega b(x)|u|^{\gamma+1}dx}.$$

Therefore we let

$$t_{\max} = t = \left[\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega b(x)|u|^{\gamma+1}dx}\right]^{\frac{1}{\gamma-p+1}}.$$

Thus h_u takes on a maximum value of $\dfrac{\gamma-p+1}{p(\gamma+1)}\left[\dfrac{(\int_\Omega |\nabla u|^p dx)^{\gamma+1}}{(\int_\Omega b(x)|u|^{\gamma+1}dx)^p}\right]^{\frac{1}{\gamma-p+1}}$ when $t = t_{\max}$.

By the Sobolev embedding: $W_0^{1,p}(\Omega) \hookrightarrow L^{\gamma+1}(\Omega)$, we have

$$\left(\int_\Omega |u|^{\gamma+1}dx\right)^{\frac{1}{\gamma+1}} \le S_{\gamma+1}\left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

where $S_{\gamma+1}$ denotes the Sobolev constant.

Hence

$$\frac{(\int_\Omega |\nabla u|^p dx)^{\gamma+1}}{(\int_\Omega |u|^{\gamma+1}dx)^p} \ge \frac{1}{S_{\gamma+1}^{p(\gamma+1)}}.$$

Thus

$$h_u(t_{\max}) \ge \frac{\gamma-p+1}{p(\gamma+1)}\left[\frac{1}{\|b^+\|_\infty^p S_{\gamma+1}^{p(\gamma+1)}}\right]^{\frac{1}{\gamma-p+1}} = \delta,$$

where δ is independent of u.

We shall now show that there exists $\lambda_1 > 0$ such that $\phi_u(t_{\max}) > 0$, i.e.,

$$h_u(t_{\max}) - \frac{\lambda(t_{\max})^{\alpha+1}}{\alpha+1}\int_\Omega a(x)|u|^{\alpha+1}dx > 0,$$

for all $u \in W - \{0\}$ provided $\lambda < \lambda_1$. We have

$$\begin{aligned}
\frac{(t_{\max})^{\alpha+1}}{\alpha+1}\int_\Omega a(x)|u|^{\alpha+1}dx &\leq \frac{1}{\alpha+1}\left[\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega b(x)|u|^{\gamma+1}dx)}\right]^{\frac{\alpha+1}{\gamma-p+1}} \\
& \|a\|_\infty S_{\alpha+1}^{\alpha+1}\left(\int_\Omega |\nabla u|^p dx\right)^{\frac{\alpha+1}{p}} \\
&= \frac{1}{\alpha+1}\|a\|_\infty S_{\alpha+1}^{\alpha+1}\left[\frac{(\int_\Omega |\nabla u|^p dx)^{\gamma+1}}{(\int_\Omega b(x)|u|^{\gamma+1}dx)^p)}\right]^{\frac{\alpha+1}{p(\gamma-p+1)}} \\
&= \frac{1}{\alpha+1}\|a\|_\infty S_{\alpha+1}^{\alpha+1}\left[\frac{p(\gamma+1)}{\gamma-p+1}\right]^{\frac{\alpha+1}{p}} h_u(t_{\max})^{\frac{\alpha+1}{p}} \\
&= c h_u(t_{\max})^{\frac{\alpha+1}{p}}
\end{aligned}$$

where c is independent of u. Hence

$$\phi_u(t_{\max}) \geq h_u(t_{\max}) - \lambda c h_u(t_{\max})^{\frac{\alpha+1}{p}} = h_u(t_{\max})^{\frac{\alpha+1}{p}}\left[h_u(t_{\max})^{\frac{p-\alpha-1}{p}} - \lambda c\right].$$

and so, since $h_u(t_{\max}) \geq \delta$ for all $u \in W - \{0\}$, it follows that

$$\phi_u(t_{\max}) \geq \delta^{\frac{\alpha+1}{p}}\left[\delta^{\frac{p-\alpha-1}{p}} - \lambda c\right].$$

Thus $\phi_u(t_{\max}) > 0$ for all non-zero u provided $\lambda < \dfrac{\delta^{\frac{p-\alpha-1}{p}}}{c} = \lambda_1$. This completes the proof. ■

It follows from the Lemma 2.7 that when $\lambda < \lambda_1$, $\int_\Omega a(x)|u|^{\alpha+1}dx > 0$ and $\int_\Omega b(x)|u|^{\gamma+1}dx > 0$ then ϕ_u must have exactly two critical points as discussed in the remarks preceding the Lemma 2.7.

Thus when $\lambda < \lambda_1$ we have obtained a complete knowledge of the number of critical points of ϕ_u, of the intervals on which ϕ_u is increasing and decreasing and of the multiples of u which lie in $M_\lambda(\Omega)$ for every possible choice of signs of $\int_\Omega a(x)|u|^{\alpha+1}dx$ and $\int_\Omega b(x)|u|^{\gamma+1}dx$. In particular we have the following result.

Corollary 2.8. $M_\lambda^0(\Omega) = \emptyset$ when $0 < \lambda < \lambda_1$.

Corollary 2.9. If $\lambda < \lambda_1$, then there exists $\delta_1 > 0$ such that $J_\lambda(u) \geq \delta_1$ for all $u \in M_\lambda^-(\Omega)$.

Proof. Consider $u \in M_\lambda^-(\Omega)$. Then ϕ_u has a positive global maximum at $t = 1$ and $\int_\Omega b(x)|u|^{\gamma+1}dx > 0$. Thus

$$\begin{aligned} J_\lambda(u) = \phi_u(1) &\geq \phi_u(t_{\max}) \geq h_u(t_{\max})^{\frac{\alpha+1}{p}}(h_u(t_{\max})^{\frac{p-\alpha-1}{p}} - \lambda c) \\ &\geq \delta^{\frac{\alpha+1}{p}}(\delta^{\frac{p-\alpha-1}{p}} - \lambda c) \end{aligned}$$

and the left hand side is uniformly bounded away from 0 provided that $\lambda < \lambda_1$. ■

3. Existence results

Now we can state our main result.

Theorem 3.1. If $\lambda < \lambda_1$, there exists a minimizer of J_λ on $M_\lambda^+(\Omega)$.

Proof. Since J_λ is bounded below on $M_\lambda(\Omega)$ and so on $M_\lambda^+(\Omega)$, there exists a minimizing sequence $\{u_n\} \subseteq M_\lambda^+(\Omega)$ such that $\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u \in M_\lambda^+(\Omega)} J_\lambda(u)$. Then by Lemma 2.4 and Rellich-Kondrachov Theorem, there exit a subsequence $\{u_n\}$ and $u_0 \in W$ such that $u_n \to u_0$ weakly in W, $u_n \to u_0$ strongly in $L^r(\Omega)$ for $1 < r < \dfrac{np}{n-p}$.

If we choose $u \in W$ such that $\int_\Omega a(x)|u|^{\alpha+1}dx > 0$, then the graph of the fibering map ϕ_u must be of one the forms shown in Figure 2(b) or (d) and so there exists $t_1(u)$ such that $t_1(u)u \in M_\lambda^+(\Omega)$ and $J_\lambda(t_1(u)u) < 0$. Hence, $\inf_{u \in M_\lambda^+(\Omega)} J_\lambda(u) < 0$. By (2.2),

$$J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{\gamma+1}\right)\int_\Omega |\nabla u_n|^p dx - \lambda\left(\frac{1}{\alpha+1} - \frac{1}{\gamma+1}\right)\int_\Omega a(x)|u_n|^{\alpha+1}dx,$$

and so

$$\lambda\left(\frac{1}{\alpha+1} - \frac{1}{\gamma+1}\right)\int_\Omega a(x)|u_n|^{\alpha+1}dx = \left(\frac{1}{p} - \frac{1}{\gamma+1}\right)\int_\Omega |\nabla u_n|^p dx - J_\lambda(u_n).$$

Letting $n \to \infty$, we see that $\int_\Omega a(x)|u_0|^{\alpha+1}dx > 0$.

Suppose $u_n \not\to u_0$ in W. We shall obtain a contradiction by discussing the fibering map. Since $\int_\Omega a(x)|u_0|^{\alpha+1}dx > 0$, the graph of ϕ_{u_0} must be either of the from shown in Figure 2(b) or (d). Hence there exists $t_0 > 0$ such that $t_0u_0 \in M_\lambda^+(\Omega)$ and ϕ_{u_0} is decreasing on $(0, t_0)$ with $\phi'_{u_0}(t_0) = 0$.

Since $u_n \not\to u_0$ in W, then

$$\|u_0\| < \lim_{n\to\infty}\inf \|u_n\| \Rightarrow \int_\Omega |\nabla u_0|^p dx < \lim_{n\to\infty}\inf \int_\Omega |\nabla u_n|^p dx.$$

Thus, as

$$\phi'_{u_n}(t) = t^p \int_\Omega |\nabla u_n|^p dx - \lambda t^\alpha \int_\Omega a(x)|u_n|^{\alpha+1} dx - t^\gamma \int_\Omega b(x)|u_n|^{\gamma+1} dx,$$

and

$$\phi'_{u_0}(t) = t^p \int_\Omega |\nabla u_0|^p dx - \lambda t^\alpha \int_\Omega a(x)|u_0|^{\alpha+1} dx - t^\gamma \int_\Omega b(x)|u_0|^{\gamma+1} dx.$$

Since $\{u_n\}$ tends to u_0 strongly in L^r, we have

$$\begin{aligned} 0 &= \phi'_{u_0}(t_0) = t_0^{p-1} \int_\Omega |\nabla u_0|^p dx - \lambda t_0^\alpha \int_\Omega a(x)|u_0|^{\alpha+1} dx - t_0^\gamma \int_\Omega b(x)|u_0|^{\gamma+1} d(x) \\ &< \lim_{n\to\infty} \inf \left(t_0^{p-1} \int_\Omega |\nabla u_n|^p dx - \lambda t_0^\alpha \int_\Omega a(x)|u_n|^{\alpha+1} dx - t_0^\gamma \int_\Omega b(x)|u_n|^{\gamma+1} d(x) \right) \\ &= \liminf n \to \infty \phi'_{u_n}(t_0). \end{aligned}$$

It follows that $\phi'_{u_n}(t_0) > 0$ for n sufficiently large. Since $\{u_n\} \subseteq M_\lambda^+(\Omega)$, by considering the possible fibering maps it is easy to see that $\phi'_{u_n}(t) < 0$ for $0 < t < 1$ and $\phi'_{u_n}(1) = 0$ for all n. Hence we must have $t_0 > 1$. But $t_0 u_0 \in M_\lambda^+(\Omega)$ and so

$$J_\lambda(t_0 u_0) = \phi_{u_0}(t_0) < \phi_{u_0}(1) = J_\lambda(u_0) < \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u \in M_\lambda^+} J_\lambda(u).$$

and this is a contradiction. Hence $u_n \to u_0$ in W and so

$$J_\lambda(u_0) = \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u \in M_\lambda^+} J_\lambda(u).$$

Thus u_0 is a minimizer for J_λ on $M_\lambda^+(\Omega)$. ■

Theorem 3.2. If $\lambda < \lambda_1$, there exists a minimizer of J_λ on $M_\lambda^-(\Omega)$.

Proof. By Corollary 2.9 we have $J_\lambda(u) \geq \delta_1 > 0$ for all $u \in M_\lambda^-(\Omega)$ and so $\inf_{u \in M_\lambda^-(\Omega)} J_\lambda(u) > 0$. Hence there exists a minimizing sequence $\{u_n\} \subseteq M_\lambda^-(\Omega)$ such that

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u \in M_\lambda^-(\Omega)} J_\lambda(u) > 0.$$

As in the previous proof, since J_λ is coercive, $\{u_n\}$ is bounded in W and we may assume, without loss of generality, that $u_n \to u_0$ weakly in W, $u_n \to u_0$ strongly in $L^r(\Omega)$ for $1 < r < \dfrac{np}{n-p}$. By (2.2)

$$J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{\alpha+1}\right) \int_\Omega |\nabla u_n|^p dx + \left(\frac{1}{\alpha+1} - \frac{1}{\gamma+1}\right) \int_\Omega b(x)|u_n|^{\gamma+1} dx.$$

and, since $\lim_{n\to\infty} J_\lambda(u_n) > 0$ and $\lim_{n\to\infty} \int_\Omega b(x)|u_n|^{\gamma+1}dx = \int_\Omega b(x)|u_0|^{\gamma+1}dx$ we must have that $\int_\Omega b(x)|u_0|^{\gamma+1}dx > 0$. Hence the fibering map ϕ_{u_0} must have graph as shown in Figure 2(c) or (d) and so there exists $\hat{t} > 0$ such that $\hat{t}u_0 \in M_\lambda^-(\Omega)$.

Suppose $u_n \not\to u_0$ in W. Using the facts that

$$\int_\Omega |\nabla u_0|^p dx < \lim_{n\to\infty} \inf \int_\Omega |\nabla u_n|^p dx,$$

and that, since $u_n \subseteq M_\lambda^-(\Omega)$, $\phi_{u_n}(1) = J_\lambda(u_n) \geq J_\lambda(su_n) = \phi_{u_n}(s)$, for all $s \geq 0$, we have

$$\begin{aligned} J_\lambda(\hat{t}u_0) &= \frac{1}{p}\hat{t}^p \int_\Omega |\nabla u_0|^p dx - \frac{\lambda \hat{t}^{\alpha+1}}{\alpha+1}\int_\Omega a(x)|u_0|^{\alpha+1}dx - \frac{\hat{t}^{\gamma+1}}{\gamma+1}\int_\Omega b(x)|u_0|^{\gamma+1}dx \\ &< \lim_{n\to\infty}\left[\frac{1}{p}\hat{t}^p \int_\Omega |\nabla u_n|^p dx - \frac{\lambda \hat{t}^{\alpha+1}}{\alpha+1}\int_\Omega a(x)|u_n|^{\alpha+1}dx - \frac{\hat{t}^{\gamma+1}}{\gamma+1}\int_\Omega b(x)|u_n|^{\gamma+1}dx\right] \\ &= \lim_{n\to\infty} J_\lambda(\hat{t}u_n) \\ &\leq \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u). \end{aligned}$$

which is a contradiction. Hence $u_n \to u_0$ in W and the proof can be completed as in the previous Theorem. ■

Corollary 3.3. Equation (1.1) has at least two positive solutions whenever $0 < \lambda < \lambda_1$.

Proof. By Theorems 3.1 and 3.2 there exist $u^+ \in M_\lambda^+(\Omega)$ and $u^- \in M_\lambda^-(\Omega)$ such that $J_\lambda(u^+) = \inf_{u\in M_\lambda^+(\Omega)} J_\lambda(u)$ and $J_\lambda(u^-) = \inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u)$.

Moreover $J_\lambda(u^\pm) = J_\lambda(|u^\pm|)$ and $|u^\pm| \in M_\lambda^\pm(\Omega)$ and so we may assume $u^\pm \geq 0$. By Lemma 2.3 $u^\pm$ are critical points of J_λ on W and hence are weak solutions (and so by standard regularity results classical solutions) of (1.1). Finally, by the Harnack inequality due to Trudinger [9], we obtain that $u^\pm$ are positive solutions of (1.1). ■

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