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Design of a Optimal Wave Reflector to Protect the Beaches: (A Mathematical Modelling Approach)

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Abstract

One possible way to protect the beaches from incoming waves would be to try to partially reflect the waves by constructing suitable bottom variations in the sea in front of the beach. To investigate the viability of this basic idea is our aim of this modeling approach. The basic assumption that we will make is that the gravity driven surface waves are considered to be of low amplitude, which allows us to restrict to linear waves. Furthermore, assuming shallow water, dispersive effects are neglected which means that we assume that each sinusoidal wave travels with a velocity that depends only on the depth of the water.

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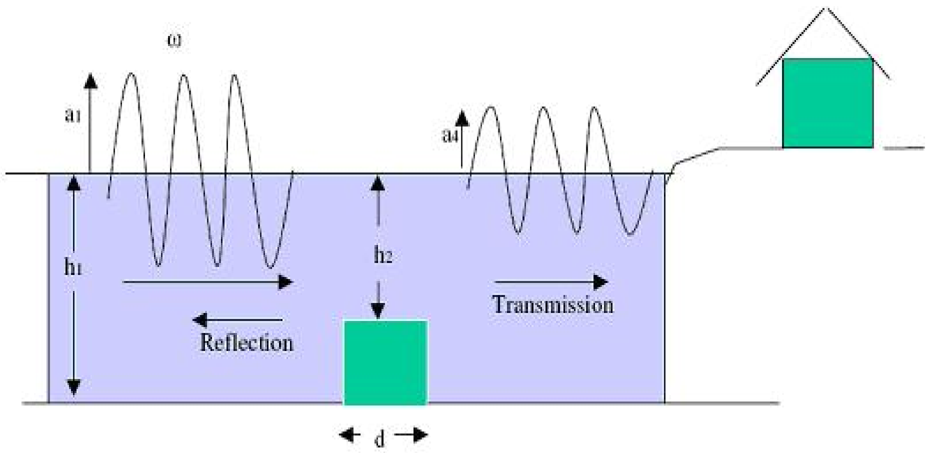

Figure 1:

1. Introduction

Due to love of nature and natural scenery people construct their homes, hotels , resorts and develop beaches on wonderful sea shores, which are considered to be premium properties in real estate. A great loss of this valuable properties and land occurs due to erosion of land due to sea waves over the years. People are always bothered about devising feasible ways to get protection from the sea waves. One such possible way to protect the beaches from incoming waves would be to try to partially reflect the waves by constructing suitable bottom variations in the sea in front of the beach as depicted in Figure 1. To investigate the viability of this basic idea was the aim of this modeling assignment. One investigation could direct towards placing a bar of some material perpendicular to the direction of the wave propagation. We will discuss the above problems under three sections.

(A) The One Bar Problem

1.1. Problem Description and basic assumptions:

Though the full investigation of this physical situation may be very complicated, a simplified model still provides valuable information and near reality approximations. Which in turn may be used to find effects and optimized dimensions of such partial wave reflectors. To get a simplistic, solvable but near realistic model, the following assumptions were made.

The waves were considered to be gravity driven surface waves. Such types of waves are considered to be low amplitude and can be considered as linear waves. Further more, assuming shallow water, the dispersive effects are neglected. That is we assume that each sinusoidal wave travels with velocity that depends only on the depth of the sea water

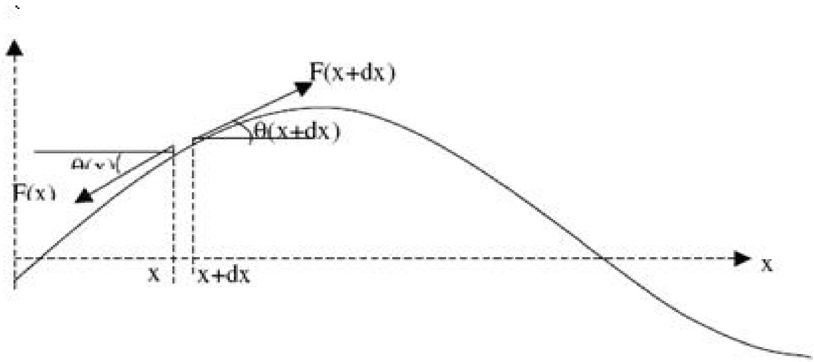

Figure 2:

denoted as h_1, the amplitude and frequency of the waves incoming from the deep sea, a_1 and ω. The waves are considered to be monochromatic, that is waves are of uniform frequency. Problem thus boiled down to investigating the reduction of the outgoing wave amplitude, a_4 and its dependence on the various parameters like dimensions of bar width d and water column above the bar h_2 as shown in Figure 1. Thus the reflection-transmission properties of the incoming waves was investigated with severely simplified model.

Further the waves were assumed to be purely gravity driven - the so called "Gravity waves" whence the speed of the waves was assumed to be dependent on the acceleration due to gravity "g" and the depth of the water from a still-water level as

$$c \propto \sqrt{gh}$$

where c is the speed of the wave in a medium g is the acceleration due to gravity and h is the depth of water.

However, some dimensional analysis was done, and, by the principle of homogeneity, the assumption was found to be fairly reasonable, The possible The possible dependence on an additional list of quantities like surface tension, coefficient of viscosity, ambient pressure & temperature, vapour pressure above the liquid surface, wavelength, distance of moon, time of the day of observation, amplitudes of the wave, angular speed of the earth, etc as considered for the purpose.

The following were the quantities that are known in the problem. The amplitude and frequency of the waves incoming from the deep sea, a_1 and ω, the height of the still water level h_1, the height of the water level above the bar, h_2, and the width of the bar d. The problem thus boiled down to investigating the reduction of the outgoing wave amplitude, a_4 and its dependence on the various parameters detailed above.

2. Modeling of the water waves

A small mass element on the surface of the wave executes harmonic motion with the displacement y, as shown in the figure above, depending on the position x and the time t as,

$$y = y(x, t) \tag{2.1}$$

The force and angle θ made by the tangents to the mass elements with the direction of the propagation differs and the equation of motion gives

$$[F sin\theta](x + dx) = dm\frac{\partial^2 y}{\partial t^2} \tag{2.2}$$

where $dm = \rho dx$, and ρ is the linear mass density. This reduces to the differential equation

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{2.3}$$

where $c^2 = F/\rho$ This second order partial differential equation so obtained could be variable separable to give two second order ordinary differential equations from

$$c^2\frac{X^{''}}{X} = \frac{1}{c^2}\frac{T^{''}}{T} = -k^2(say) \tag{2.4}$$

for a choice of common ratio to be negative to yield physically relevant solution. Thus one gets,

$$y = (Ae^{ikx} + Be^{-kx})(Ce^{i\omega t} + De^{-i\omega t}) \tag{2.5}$$

$$y = [ACe^{i(kx+\omega t)} + BDe^{-i(kx+\omega t)}] + [ADe^{i(kx-\omega t)} + BCe^{-i(kx-\omega t)} \tag{2.6}$$

where $exp[i(kx-\omega t)]$ represents the wave traveling in the $+x$ direction and $exp[-i(kx-\omega t)]$ represent the wave traveling in the backward direction.

3. Wave equations and their solutions in this problem

The problem can be thus considered analogous to the optical problem of a wave propagating through a rarer medium in which a denser medium has been inserted. The wave thus propagates through different media with different wave velocities. The wave equations in the different regions can be written as

Region number I: The speed of the wave being c_1,

$$c_1{}^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{3.1}$$

$$y_1 = a_1 e^{i(k_1 x-\omega t)} \tag{3.2}$$

$$y_3 = a_3 e^{-i(k_1 x+\omega t)} \tag{3.3}$$

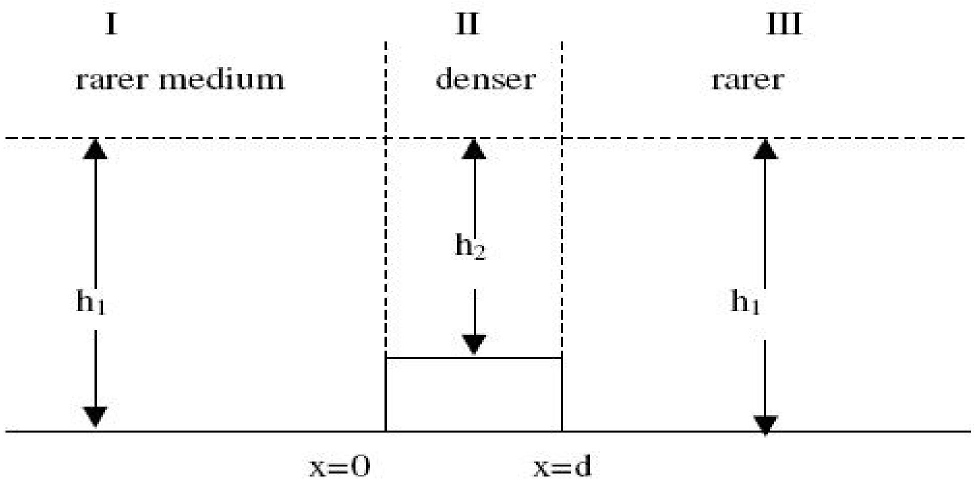

Figure 3:

$$y_I = y_1 + y_3$$

Region number II: The speed of the wave being c_2,

$$c_2{}^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{3.4}$$

$$y_2 = a_2 e^{i(k_2 x - \omega t)} \tag{3.5}$$

$$y_5 = a_5 e^{-i(k_2 x + \omega t)} \tag{3.6}$$

$$y_{II} = y_2 + y_5$$

Region number III: The speed of the wave being c_1,

$$c_1{}^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{3.7}$$

$$y_4 = a_4 e^{ip(x - c_1 t)} \tag{3.8}$$

$$y_{III} = y_4$$

The wave equations: The wave equations are thus obtained to be

$$\begin{aligned} y_I &= a_1 e^{i(k_1 x - \omega t)} + a_3 e^{-i(k_1 x + \omega t)} \\ y_{II} &= a_2 e^{i(k_2 x - \omega t)} + a_5 e^{-i(k_2 x + \omega t)} \\ y_{III} &= a_4 e^{ip(x - c_1 t)} \end{aligned}$$

4. Boundary Conditions

The boundary conditions at the “interfaces” were fixed. The continuity of the wave surface and of the energy flux were found to be two good boundary conditions at each “interface” x = 0 was taken as shown in the figure above. Thus they are,

$$y_I = y_{II} \quad at\ x = 0 \tag{4.1}$$

$$y_{II} = y_{III} \quad at\ x = d \tag{4.2}$$

$$y_I = y_{II} \quad at\ x = d \tag{4.3}$$

$$\left[{c_2}^2 \frac{\partial y}{\partial x}\right]_I = \left[{c_2}^2 \frac{\partial y}{\partial x}\right]_{II} \tag{4.4}$$

$$\left[{c_2}^2 \frac{\partial y}{\partial x}\right]_{II} = \left[{c_2}^2 \frac{\partial y}{\partial x}\right]_{III} \tag{4.5}$$

Solutions:
Applying the boundary conditions, the relation along the amplitude was found to be

$$a_1 + a_3 = a_2 + a_5 \tag{4.6}$$

$$a_2 e^{ip_2 d} + a_5 e^{-ip_2 d} = a_4 e^{ip_1 d} \tag{4.7}$$

$$c_1(a_1 - a_3) = c_2(a_2 - a_5) \tag{4.8}$$

$$c_2(a_2 e^{ip_2 d} - a_5 e^{-ip_2 d}) = c_1 a_4 e^{ip_1 d} \tag{4.9}$$

5. Concluding Remarks

The above equations were solved to get an expression for the amplitude of the outgoing wave in relation to the amplitude of the incoming wave.

$$a_4 = \frac{4c_1c_2}{(c_1 - c_2)^2 + (c_1 + c_2)^2 e^{-2ip_2 d}} e^{-i(p+p_2)d} a_1 \tag{5.1}$$

For a homogeneous and isotropic cosmological model, the space-time metric is given by

$$ds^2 = c^2 dt^2 - R^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right] \tag{5.2}$$

This defines the space-time geometry of the universe on large scale co-moving coordinates. The time ‘t’ is the proper time for each particle, k is the curvature index such that $k = 1$ for a close model, $k = 0$ for a flat model and $k = -1$ for an open model, R(t) is the scale factor.

$$T^{ij} = \left(\frac{p}{c^2} + \rho\right) u^i u^j - pg^{ij} \tag{5.3}$$

where p is the proper pressure, ρ is the proper density and w is the four-velocity vectors of the fluid particles. Since the particles are at rest in the co-moving coordinate system we have

$$u^i = (0, 0, 0, c) \tag{5.4}$$

Substituting the values from (5), (6) & (7) in (1) we get the following set of equations:

$$\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{kc^2}{R^2} = \frac{8\pi G\rho}{c^2} - \Lambda c^2 \quad \& \quad \frac{3\dot{R}^2}{R^2} + \frac{3kc^2}{R^2} = 8\pi G\rho + \Lambda c^2 \tag{5.5}$$

Adding above equations and after simplification we get

$$c^2\frac{d}{dt}(\rho R^3) = -3pR^2\dot{R}, \quad 8\pi\dot{G}\rho + \dot{\Lambda}c^2 = 0$$
$$\frac{3\ddot{R}}{R} = -4\pi GR\left[\frac{3p}{c^2} + \rho - \frac{\Lambda c^2}{4\pi G}\right]. \tag{5.6}$$

The above equations are the fundamental equations governing the cosmological models, provided the dependence of p on ρ, and dependence of G on Λ. We consider the dependence of p on ρ in a homogeneous & isotropic model as $p = \beta\rho c^2$, where β is a positive constant. For $\beta = 0$, $p = 0$ and then the model is pressure less , for $\beta = \frac{1}{3}$, $3p = \rho c^2$ and then the model is filled with radiation. The equation(9) implies that $\dot{G}$ & $\dot{\Lambda}$ are of opposite sign.

Following Berman and Rahman we assume the variation of $G\rho$ and Λ as case first: $8\pi G\rho = At^{-n}$, $\Lambda c^2 = -Bt^{-n}$. where $A > 0$ and $B > 0$, By using above equations, after some simplification we get

$$8\pi G\dot{\rho} = n(B - A)t^{-n-1} \tag{5.7}$$

The above equation gives $\frac{\dot{\rho}}{\rho} = \frac{n(B - A)}{A}\frac{1}{t}$, which on integration under the condition $\rho = \rho_0$ at $t = t_0$ gives

$$\frac{n^2(A - B)^2}{3A^2(1 + \beta)^2}\frac{1}{t^2} + \frac{3kc^2 t_0^{\frac{2n(A-B)}{3(1+\beta)A}}}{R_0^2 t^{\frac{2n(A-B)}{3(1+\beta)A}}} = \frac{A}{t^n} - \frac{B}{t^n} \tag{5.8}$$

The only possibility under which the equation (11) is satisfied is k=o, n=2 and $4(A-B) = 3A^2(1 + \beta)^2$.

5.1. Cosmological Solutions and "big-rip"

In this section we will try to investigate the solutions to the Friedmann equations that corresponds to a quintessence scalar field θ assuming an equation of state with perfect fluid form [4]. $p = \xi\rho$ where, the parameter ξ is assumed to be constant and may be any value with in the interval $\left(-\frac{3}{2}, -\frac{2}{3}\right)$. The pressure p and energy density ρ is defined as : $p = \frac{1}{2}\dot{\xi}^2 - E(\xi)$ & $\rho = \frac{1}{2}\dot{\xi}^2 + E(\xi)$ with $E(\xi)$ the potential energy for quintessence field ξ. With the help of the above equation and the general equation of cosmic conservation, we obtain for the energy density $\rho = IR^{-3(1+\xi)}$ where I is an integration constant which should be real and positive. Here if we consider the case in which the cosmic vacuum dark energy only contains a quintessence field ξ then with the help of Friedmann equations we have $\left(\frac{\dot{R}}{R}\right)^2 = A(R)^{-3(1+\xi)}$ with $A = \left(\frac{8\pi GI}{3}\right)$.

After integration we get

$$R(t) = \left[{R_0}^{3(1+\xi)/2} + \frac{3(1+\xi)\sqrt{A}}{2}(t - t_0)\right]^{2/3(1+\xi)} \quad (5.9)$$

where R_0 and t_0 are initial radius and time respectively.

For $\xi > -1$ this solution describes an accelerating Universe whose scale factor increases towards infinity as $t \to \infty$. The case $\xi < -1$ is the case of so called phantom dark energy for which the dominant energy condition violated [5], $p + \rho < 0$, the energy density is surprisingly increasing so after simplification it is easy to see that the scale factor blows up at a finite time rises "big-rip".

6. Concluding Remarks

Here in this paper, the only possibility under which the equation (11) is satisfied is k=o, n=2 and $4(A - B) = 3A^2(1 + \beta)^2$, if we deal with the big-rip scenario as suggested by Caldwell and Weinberg according to which the universe is inexorably fated to a doomsday if $\xi > -1$. However, there could have a way out from this. In fact the solution describes in equation shows two branches around the critical time, one along which the expansion of the universe dramatically accelerates towards the singularity at the critical time and other for $t > t_c$. that describes a universe which exponentially decelerates towards zero size as $t \to \infty$.

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