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Quantum Algorithm for Sort Problem

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Abstract

A quantum algorithm for the sort problem and its example are reported. When n natural numbers that are arranged at random and may be included some parts of same them are arranged in order of size, a computational complexity of a classical computation is $n(n-1)/2[= Z]$. In the quantum algorithm by using quantum phase inversion gates and quantum inversion about mean gates, its computational complexity is about $n^{3/2}[= S]$. Therefore, a high-speed process becomes possible because S/Z is about $1/n^{1/2}$.

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1. Introduction

A quantum computer can solve a problem at high speed by a parallel computation that uses quantum entangled states. Deutsch-Jozsa's algorithm for the rapid solution [1–3], Shor's algorithm for the factorization [2–4], Grover's algorithms for the database search [2,5–7], Durr-Hoyer's algorithm for finding a minimum [8], and so on are known. A quantum algorithm for the knapsack problem has recently been reported by Fujimura [9]. The sort problem [10] is examined this time to expand the application range of the quantum computation. Therefore, its result is reported.

2. Sort problem

When n natural numbers that are arranged at random and may be included some parts of same them are arranged in order of size, a computational complexity of a classical computation is $n(n-1)/2$.

3. Quantum algorithm

It is assumed that n natural numbers which are arranged at random and may be included some parts of same them are $x_0, x_1, \ldots, x_{n-2}$, and x_{n-1}, and they are arranged from the maximum to the minimum. (When they are arranged from the minimum to the maximum, an inequality is changed as will be seen later.) First of all, the quantum registers $|a_0>, |a_1>, \ldots, |a_{n-1}>$, and $|b>$ are prepared. When α is a minimum integer that is $log_2 n$ or more, each of $|a_i>$ that i is an integer from 0 to $n-1$ is consisted of α quantum bits (=qubits). States of $|a_i>$ and $|b>$ are a_i and b, respectively.

Step 1: Each qubit of $|a_i>$ and $|b>$ is set $|0>$.

Step 2: The Hadamard gate $\boxed{\mathrm{H}}$ [2, 3] acts on each qubit of $|a_i>$. It changes them for entangled states. The total states are $(2^\alpha)^n$.

Step 3: It is assumed that a quantum gate (A) changes $|b>$ for $|b+1>$ at $a_i \geq n$, or it doesn't change $|b>$ at $a_i < n$. As a target state for $|b>$ is 0, quantum phase inversion gates (PI) and quantum inversion about mean gates (IM) [2, 5–7] act on $|b>$. When β is a minimum even integer that is $(2^\alpha/n)^{1/2}$ or more, the total number that (PI) and (IM) act on $|b>$ is β because they are a couple. Next, an observation gate (OB) observes $|b>$. These actions are repeated sequentially from $|a_0>$ to $|a_{n-1}>$. Therefore, each state of $|a_i>$ is $0, 1, \ldots, n-2$, or $n-1$, and the total states become $n^n[= W_0]$.

Step 4: At a quantum gate (B), states of $|a_0>$ and $|a_1>$ are read, and x_{a_0} and x_{a_1} are compared. In case of $x_{a_0} + (a_1 - a_0)/(n-1) > x_{a_1}$, $|b>$ isn't changed. For example, at $x_{a_u} = x_{a_{u+1}}$ and $a_{u+1} > a_u$ ($u = 0, 1, \ldots, n-2.u$ is an integer.), an arrangement of an answer is x_{a_u} and $x_{a_{u+1}}$, but at $x_{a_u} = x_{a_{u+1}}$ and $a_u > a_{u+1}$, it is $x_{a_{u+1}}$ and x_{a_u}. (When sizes of $x_0, x_1, \ldots, x_{n-2}$, and x_{n-1} are arranged from the minimum to the maximum, the inequality is $x_{a_u} - (a_{u+1} - a_u)/(n-1) < x_{a_{u+1}}$.) In case of $x_{a_0} + (a_1 - a_0)/(n-1) \leq x_{a_1}$, (B) changes $|b>$ for $|b+1>$. As a target state for $|b>$ is 0, (PI) and (IM) act on $|b>$. A frequency of (PI) and (IM) is a minimum even integer γ_0 that is $(W_0/W_1)^{1/2} = (2n/(n-1))^{1/2} \leq \gamma_0$ because a number of combinations of a_i in this condition becomes $W_1 = (n(n-1)/2)n^{n-2}$. Where, W_k is $((n-(k-1))(n-k)/2)n^{n-(k+1)}$ in $k = 1, \ldots, n-3$, and $n-2$ (k is an integer). After this, only $|b>$ is observed by (OB), and the combinations of W_1 remain. Similarly, the following operation is repeated in $k = 1, \ldots, n-3$, and $n-2$ (k is an integer). In a word, at (B), states of $|a_k>$ and $|a_{k+1}>$ are read, and x_{a_k} and $x_{a_{k+1}}$ are compared. In case of $x_{a_k} + (a_{k+1} - a_k)/(n-1) > x_{a_{k+1}}$, $|b>$ isn't changed. In case of $x_{a_k} + (a_{k+1} - a_k)/(n-1) \leq x_{a_{k+1}}$, (B) changes $|b>$ for $|b+1>$. As a target state for $|b>$ is 0, (PI) and (IM) act on $|b>$. The frequency of (PI) and (IM) is the minimum even integer γ_{k+1} that is $(W_k/W_{k+1})^{1/2} = (n(n-(k-1))/(n-(k+1)))^{1/2} \leq \gamma_{k+1}$ because the number of combinations of $|a_i>$ becomes $W_{k+1} = ((n-k)(n-(k+1))/2)n^{n-((k+1)+1)}$. After this, only $|b>$ is observed by (OB), and the combinations of W_{k+1} remain. Where, at the last (OB), $|a_i>$ and $|b>$ are observed. Therefore, it is obtained that the combination of states of $|a_i>$ is arranged from the maximum to the minimum in n natural numbers.

4. Numerical calculation

One example is given here. It is assumed that 100 natural numbers are $x_0 = 52, x_1 = 9, \ldots, x_{12} = 75, \ldots, x_{26} = 2, \ldots, x_{37} = 99, \ldots, x_{61} = 1, \ldots, x_{83} = 75, \ldots, x_{98} = 100$, and $x_{99} = 73$ ($x_{0,\ldots,99}$: $1, \ldots, 73, 75, 75, \ldots, 100$, at random), and they are arranged from the maximum to the minimum. First of all, $|a_0>, |a_1>, \ldots, |a_{99}>$, and $|b>$ are prepared. Because of $log_2 100 \approx 6.6 \leq 7 = \alpha$, each of $|a_0>, |a_1>, \ldots, |a_{98}>$, and $|a_{99}>$ is consisted of seven qubits. States of $|a_0>, |a_1>, \ldots, |a_{99}>$, and $|b>$ are $a_0, a_1, \ldots, a_{99}$, and b, respectively.

Step 1: Each qubit of $|a_0>, |a_1>, \ldots, |a_{99}>$, and $|b>$ is set $|0>$.

Step 2: $\boxed{\text{H}}$ acts on each qubit of $|a_0>, |a_1>, \ldots, |a_{98}>$, and $|a_{99}>$. It changes them for entangled states. The total states are $(2^7)^{100} \approx 6.2 \times 10^{210}$.

Step 3: (A) changes $|b>$ for $|b+1>$ at $a_0 \geq 100$, or it doesn't change $|b>$ at $a_0 < 100$. As the target state for $|b>$ is 0, (PI) and (IM) act on $|b>$. Because of $(2^7/100)^{1/2} \approx 1.1 \leq 2$, the total number that they act on $|b>$ is 2. Next, (OB) observes $|b>$. By these actions, the state of $|a_0>$ is $0, 1, \ldots, 98$, or 99. Moreover, these actions are repeated sequentially from $|a_1>$ to $|a_{99}>$. Therefore, the total states become $100^{100} = 10^{200}[= W_0]$.

Step 4: At (B), states of $|a_0>$ and $|a_1>$ are read, and x_{a_0} and x_{a_1} are compared. In case of $x_{a_0} + (a_1 - a_0)/99 > x_{a_1}$, $|b>$ isn't changed. (When sizes of $x_0, x_1, \ldots, x_{98}$, and x_{99} are arranged from the minimum to the maximum, the inequality is $x_{a_u} - (a_{u+1} - a_u)/99 < x_{a_{u+1}}$.) The frequency of (PI) and (IM) is $2 (= \gamma_1)$ that is $(W_0/W_1)^{1/2} = (2 \cdot 100/99)^{1/2} \approx 1.4 \leq 2$ because of $W1 = (100 \cdot 99/2)100^{98} \approx 5.0 \times 10^{199}$. After this, only $|b>$ is observed by (OB), and the combinations of $W_1 \approx 5.0 \times 10^{199}$ remain. Similarly, the following operation is repeated in $k = 1, \ldots, 97$, and 98 (k is an integer). In a word, at (B), states of $|a_k>$ and $|a_{k+1}>$ are read, and x_{a_k} and $x_{a_{k+1}}$ are compared. In case of $x_{a_k} + (a_{k+1} - a_k)/99 > x_{a_{k+1}}$, $|b>$ isn't changed. In case of $x_{a_k} + (a_{k+1} - a_k)/99 \leq x_{a_{k+1}}$, (B) changes $|b>$ for $|b+1>$. As a target state for $|b>$ is 0, (PI) and (IM) act on $|b>$. The frequency of (PI) and (IM) is γ_{k+1} that is $(W_k/W_{k+1})^{1/2} = (100(100-(k-1))/(100-(k+1)))^{1/2} \leq \gamma_{k+1}$ because the number of combinations of $|a_i>$ becomes $W_{k+1} = ((100-k)(100-(k+1))/2)100^{100-((k+1)+1)}$. After this, only $|b>$ is observed by (OB), and the combinations of W_{k+1} remain. By the way, when a condition is $a_{25} = 12$ and $a_{26} = 83$, the inequality is $x_{a_{25}} + (a_{26} - a_{25})/99 = x_{12} + (83 - 12)/99 = 75 + 71/99 \approx 75.7 > x_{a_{26}} = x_{83} = 75$. Therefore, the arrangement of the answer is x_{12} and x_{83}. And, the values of $\gamma_2, \gamma_3, \ldots, \gamma_{98}$, and γ_{99} are $\gamma_2 = \gamma_3 = \ldots = \gamma_{95} = 12, \gamma_{96} = \gamma_{97} = 14, \gamma_{98} = 16$, and $\gamma_{99} = 18$. Moreover,

W_{99} is 1. Therefore, at the last (OB), $|a_i>$ and $|b>$ are observed. The states of $|a_0>$, $|a_1>, \ldots, |a_{25}>, |a_{26}>, |a_{27}>, \ldots, |a_{48}>, \ldots, |a_{91}>, \ldots, |a_{98}>, |a_{99}>$, and $|b>$ are $98, 37, \ldots, 12, 83, 99, \ldots, 0, \ldots, 1, \ldots, 26, 61$, and 0, respectively. Finally, the answer is $x_{98} = 100, x_{37} = 99, \ldots, x_{12} = 75, x_{83} = 75, x_{99} = 73, \ldots, x_0 = 52, \ldots, x_1 = 9, \ldots, x_{26} = 2$, and $x_{61} = 1$.

5. Discussion and Summary

The computational complexity of this quantum algorithm ($= S$) becomes the following. In the order of the actions by the gates, the number of them is $n\alpha$ at $\boxed{\text{H}}$, n at (A), $2n$ at (PI) and (IM), n at (OB), $2(n-1)$ at (B), $\Sigma_{f=1}^{n-1}\gamma_f$ at (PI) and (IM), and $n-1$ at (OB). Therefore, S becomes $(\alpha+4)n + 3(n-1) + \Sigma_{f=1}^{n-1}\gamma_f$. In the example of the section 4, S is 2589. It corresponds to about $1/2$ of the computational complexity of the classical calculation ($= Z$) because Z is 4950. When n is large enough, S becomes about $(log_2 n + 4)n + 3(n-1) + (2n/(n-1))^{1/2} + \Sigma_{f=2}^{n-1}(n(n-(f+2))/(n-f))^{1/2} \approx n \cdot log_2 n + 3n + n \cdot n^{1/2} \approx (log_2 n + 3 + n^{1/2})n \approx n^{1/2}n = n^{3/2}$, and Z becomes $n(n-1)/2 \approx n^2$. Therefore, S/Z is about $1/n^{1/2}$. For example, when n is 10^{100}, S/Z becomes about $1/10^{50}$.

Further development of the quantum computation in the future is expected.

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