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New Modular Relations and General Formulas for Explicit Evaluations of a Continued Fraction of Order Six

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Abstract

In this paper, we prove some new modular relations for a continued fraction $X(q)$ of order six by establishing new theta-function identities. We also offer general formulas for the explicit evaluations of $X(q)$ and by parameterizations.

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1. Introduction

For $|q| < 1$, Ramanujan's three special theta-functions are defined by

$$\phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (-q^2; q^2)_\infty}, \tag{1.1}$$

$$\psi(q) := f(q, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{1.2}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty. \tag{1.3}$$

where $(a; q)_\infty := \prod_{k=1}^{\infty}(1 - aq^{k-1})$. If $q = e^{2\pi i z}$ with $\text{Im}(z) > 0$, then $f(-q) = q^{-1/24}\eta(z)$, where $\eta(z)$ denotes the classical Dedekind eta-function.

Ramanujan recorded many continued fractions and some of their explicit values in his second notebook [6] and in his lost notebook [7]. Some of the Ramanujan's continued fractions can be expressed in terms of Ramanujan's theta-functions. One such continued fraction is $X(q)$ which is studied by Vasuki et al. [8] and is defined by

$$X(q) := \frac{q^{-1/4}(1-q^2)}{1-q^{3/2}} + \frac{(1-q^{1/2})(1-q^{7/2})}{q^{1/2}(1-q^{3/2})(1+q^3)} + \frac{(1-q^{5/2})(1-q^{13/2})}{q^{3/2}(1-q^{3/2})(1+q^6)} + \cdots |q| < 1. \tag{1.4}$$

In terms of Ramanujan's theta-functions, $X(q)$ [8, p. 80, (1.9)] can be expressed as

$$X(q) := q^{1/4}\frac{\psi(q^3)}{\psi(q)}. \tag{1.5}$$

Vasuki et al. [8] also established modular relations connecting the continued fraction $X(q)$ with the continued fractions $X(q^2)$, $X(q^3)$, $X(q^5)$, $X(q^7)$, and $X(q^{11})$. In this paper, we prove new modular relations connecting $X(q)$ and the continued fractions $X(-q)$ and $X(q^4)$ by establishing some new theta-function identities. We also give a new approach to the modular relation connecting $X(q)$ and $X(q^2)$ which is established in [8].

In his notebooks, Ramanujan recorded several explicit values of theta-functions $f(-q)$, $\phi(q)$ and $\psi(q)$ which are proved by Berndt and Chan [5]. An account of these can also be found in Berndt's book [4]. Recently, Yi [9] also found many new explicit values of $\phi(q)$ and its quotients by finding explicit values of the parameters $h_{k,n}$, and $h'_{k,n}$ of theta-function $\phi(q)$ which are respectively defined by

$$h_{k,n} := \frac{\phi(q)}{k^{1/4}\phi(q^k)}; \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.6}$$

and

$$h'_{k,n} := \frac{\phi(-q)}{k^{1/4}\phi(-q^k)}, \qquad q = e^{-2\pi\sqrt{n/k}}, \tag{1.7}$$

where k and n are rational numbers. Motivated by Yi's work, Baruah and Saikia [1] introduced two parameters $g_{k,n}$ and $g'_{k,n}$ of the theta-function $\psi(q)$ which are respectively defined by

$$g_{k,n} := \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)} \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.8}$$

and

$$g'_{k,n} := \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}, \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.9}$$

where k and n are rational numbers. They also calculated several explicit values of $g_{k,n}$ and $g'_{k,n}$ and found explicit values of $\psi(q)$. In this paper, we use the parameters

$h_{k,n}$, $h'_{k,n}$,and $g'_{k,n}$ defined above to prove general formulas for explicit evaluations of $X(e^{-\pi\sqrt{n/3}})$ and $X(e^{-4\pi\sqrt{n/3}})$.

In section 2, we record some preliminary results which will be used in the subsequent sections. In section 3, we prove some new theta-function identities. In section 4, we prove modular relations connecting $X(q)$ with the continued fractions $X(-q)$ and $X(q^4)$. We also give alternate proof of modular relation connecting $X(q)$ and $X(q^2)$ which is due to Vasuki et al. [8]. Finally in section 5, we establish general formulas for explicit evaluations of $X(q)$.

To end this introduction, we define Ramanujan's modular equation. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.10}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (1.10). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. By denoting $z_n = \phi^2(q^n)$, where $q = exp(-\pi K'/K)$, $|q| < 1$, the multiplier m connecting α and β is defined by

$$m = \frac{z_1}{z_n}. \tag{1.11}$$

2. Preliminary Results

Lemma 2.1. [8, p. 80, Lemma 2.1]

$$\frac{\phi^2(-q)}{\phi^2(-q^3)} = \frac{1-3X^2(q^2)}{1+X^2(q^2)} \tag{2.1}$$

Lemma 2.2. [8, p. 81, Lemma 2.6]

$$\frac{\phi^4(-q)}{\phi^4(-q^3)} = \frac{1-9X^4(q)}{1-X^4(q)}. \tag{2.2}$$

Lemma 2.3. [3, p.122, Entry 10(i), (ii), (iii)]

$$\phi(q) = \sqrt{z_1}, \tag{2.3}$$

$$\phi(-q) = \sqrt{z_1}(1-\alpha)^{1/4}, \tag{2.4}$$

$$\phi(-q^2) = \sqrt{z_1}(1-\alpha)^{1/8}. \tag{2.5}$$

Lemma 2.4. [3, p. 123, Entry 11(i),(ii)]

$$\psi(q) = \sqrt{1/2}\sqrt{z_1}\alpha^{1/8}q^{-1/8} \tag{2.6}$$

$$\psi(q^2) = \frac{1}{2}\sqrt{z_1}\alpha^{1/4}q^{-1/4}. \tag{2.7}$$

3. Theta-Function Identities

In this section, we prove some new theta-function identities which will be used in the subsequent sections.

Theorem 3.1. If $P = \dfrac{\phi(q)}{\phi(q^3)}$ and $Q = \dfrac{\phi(-q)}{\phi(-q^3)}$ then

$$P^2 + Q^2 + P^2Q^2 = 3. \tag{3.1}$$

Proof. Transcribing P and Q using (2.3) and (2.4), we have

$$P = \sqrt{\frac{z_1}{z_3}} = \sqrt{m} \quad \text{and} \quad Q = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/4}, \tag{3.2}$$

where β has degree 3 over α. From (3.2), we have

$$Q^2 = P^2\left(\frac{1-\alpha}{1-\beta}\right)^{1/2}. \tag{3.3}$$

By [3, p. 234], we have

$$\left(\frac{1-\alpha}{1-\beta}\right)^{1/2} = \frac{3-m}{m(m+1)}. \tag{3.4}$$

Employing (3.2) and (3.4) in (3.3) and then simplifying, we complete the proof. ■

Theorem 3.2. If $P = \dfrac{\phi(-q)}{\phi(-q^3)}$ and $Q = \dfrac{\phi(-q^2)}{\phi(-q^6)}$ then

$$P^4 + Q^4P^2 + Q^4 - 3P^2 = 0. \tag{3.5}$$

Proof. Transcribing P and Q using (2.4) and (2.5), we have

$$P = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/4} \quad \text{and} \quad Q = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/8}, \tag{3.6}$$

where β has degree 3 over α. Now (3.6) implies

$$m = \frac{Q^4}{P^2}. \tag{3.7}$$

Employing (3.4) in (3.6) and simplifying, we get

$$Q^4 = \frac{m(3-m)}{m+1}. \tag{3.8}$$

Invoking (3.7) in (3.8) and simplifying we complete the proof. ■

Theorem 3.3. If $P = \dfrac{\psi(q)}{q^{1/4}\psi(q^3)}$ and $Q = \dfrac{\psi(q^2)}{q^{1/2}\psi(q^6)}$ then

$$P^4 + 3Q^2 - P^4Q^2 + Q^4 = 0. \tag{3.9}$$

Proof. Transcribing P and Q using (2.6) and (2.7), we have

$$P = \sqrt{m}\left(\frac{\alpha}{\beta}\right)^{1/8} \quad \text{and} \quad Q = \sqrt{m}\left(\frac{\alpha}{\beta}\right)^{1/4}, \tag{3.10}$$

where β has degree 3 over α. Now (3.10) gives

$$m = \frac{P^4}{Q^2}. \tag{3.11}$$

Applying (3.15) in (3.10), we get

$$P^4 = \frac{m(3+m)}{m-1}. \tag{3.12}$$

Simplifying (3.12) by invoking (3.11), we complete the proof. ∎

Theorem 3.4. If $P = \dfrac{\phi(-q)}{\phi(-q^3)}$ and $Q = \dfrac{\psi(q^2)}{q^{1/2}\psi(q^6)}$ then

$$3 + P^2 - Q^2 + P^2Q^2 = 0. \tag{3.13}$$

Proof. Transcribing P and Q using (2.4) and (2.7) and simplifying, we get

$$P^2 = m\left(\frac{1-\alpha}{1-\beta}\right)^{1/2} \quad \text{and} \quad Q = m\left(\frac{\alpha}{\beta}\right)^{1/2}, \tag{3.14}$$

where β has degree 3 over α. Also by [3, p. 234], we have

$$\left(\frac{\alpha}{\beta}\right)^{1/2} = \frac{3+m}{m(m-1)}. \tag{3.15}$$

Invoking using (3.4) and (3.15) in (3.14), we arrive at

$$m = \frac{3-P^2}{P^2+1} = \frac{3+Q^2}{Q^2-1}. \tag{3.16}$$

Simplifying last two fractions of (3.16), we complete the proof. ∎

Remark 3.5. Employing (1.5) in Theorem 3.4 and simplifying, we easily arrive at Lemma 2.1. So Theorem 3.4 and Lemma 2.1 are equivalent.

Theorem 3.6. If $P = \dfrac{\phi(q)}{\phi(q^3)}$ and $Q = \dfrac{\psi(q)}{q^{1/4}\psi(q^3)}$ then

$$P^4 + Q^4 - P^2Q^4 + 3P^2 = 0. \tag{3.17}$$

Proof. Transcribing P and Q using (2.3) and (2.6), we get

$$P = \sqrt{m} \quad \text{and} \quad Q = \sqrt{m}\left(\frac{\alpha}{\beta}\right)^{1/8} \tag{3.18}$$

where β has degree 3 over α. From (3.18), we easily deduce

$$Q^4 = P^4\left(\frac{\alpha}{\beta}\right)^{1/2}. \tag{3.19}$$

Employing (3.15) in (3.19) and simplifying, we arrive at

$$Q^4 = P^4\left(\frac{3+m}{m(m-1)}\right). \tag{3.20}$$

Invoking $m = P^2$ from (3.18) in (3.20) and simplifying, we complete the proof. ■

4. New Modular Relations for $X(q)$

In this section we prove modular relations connecting continued fraction $X(q)$ and the continued fractions $X(-q)$ and $X(q^4)$. We also give alternate approach to the modular relation connecting $X(q)$and $X(q^2)$which is due to Vasuki et al. [8].

Theorem 4.1. Let $x = X(q),\ y = X(-q),\ u = X(q^2)$ *and* $v = X(q^4)$, then

$$\begin{aligned}
&\text{(i) } x^4 - x^8 + y^4 - y^8 - 10x^4y^4 + 9x^8y^4 + 9x^4y^8 = 0,\\
&\text{(ii) } x^4 - u^2 + 3x^4u^2 + u^4 = 0,\\
&\text{(iii) } x^8 - v^2 + 8x^4v^2 - 3x^8v^2 - v^4 + 16x^4v^4 - 9x^8v^4\\
&\qquad + v^6 - 24x^4v^6 + 27x^8v^6 + v^8 = 0.
\end{aligned}$$

Proof. To prove (i), we rewrite (3.1) as

$$P^2Q^2 + Q^2 = 3 - P^2 \tag{4.1}$$

which on squaring gives

$$P^4Q^4 + Q^4 - P^4 - 9 = -2P^2(3 + Q^4). \tag{4.2}$$

Again squaring (4.2), we arrive at

$$(P^4Q^4 + Q^4 - P^4 - 9)^2 = 4P^4(3 + Q^4)^2. \tag{4.3}$$

By Lemma 2.2, we have

$$P^4 = \frac{1-9X^4(-q)}{1-X^4(-q)} \text{ and } Q^4 = \frac{1-9X^4(q)}{1-X^4(q)}. \tag{4.4}$$

Now invoking (4.4) in (4.3) and simplifying we arrive at (i). *To prove (ii)*, we employ (1.5) in Theorem 3.3 and simplify. *To prove (iii)*, rewrite (3.5) as

$$P^4 + Q^4 = P^2(3 - Q^4) \tag{4.5}$$

which on squaring gives

$$(P^4 + Q^4)^2 = P^4(3 - Q^4)^2. \tag{4.6}$$

Now by Lemma 2.2, we have

$$P^4 = \frac{1-9X^4(q)}{1-X^4(q)} \tag{4.7}$$

and by Lemma 2.1, we have

$$Q^2 = \frac{1-3X^2(q^4)}{1+X^2(q^4)}. \tag{4.8}$$

Invoking (4.7) and (4.8) in (4.6) and simplifying, we complete the proof. ■

5. General Formulas for Explicit Evaluations of $X(q)$

This section is devoted to establish some general formulas for finding explicit values of $X(e^{-\pi\sqrt{n/3}})$ and $X(e^{-4\pi\sqrt{n/3}})$.

Theorem 5.1. For $q := e^{-\pi\sqrt{n/3}}$, let

$$L_n = \frac{\phi(q)}{3^{1/4}\phi(q^3)},$$

then $X(e^{-\pi\sqrt{n/3}}) = \left(\dfrac{\sqrt{3}L_n^2 - 1}{3L_n^2\left(\sqrt{3} + L_n^2\right)}\right)^{1/4}.$

Proof. Employing (1.5) and the definition of L_n in Theorem 3.6, we get

$$P = 3^{1/4}L_n \quad \text{and} \quad Q = \frac{1}{X(q)}. \tag{5.1}$$

Invoking (5.1) in (3.17) and simplifying, we complete the proof. ■

We note that the parameter L_n is a particular case of the parameter $h_{k,n}$, where $k = 3$ defined in (1.6). It is useful to note that $h_{3,n} = h_{n,3}$ [9]. Yi [9] evaluated L_n for $n = 1, 3, 1/3, 5, 1/5, 9, 1/9, 25, 1/25$. Baruah and Saikia [2] evaluated L_n for $n = 2, 1/2$.

It is clear from Theorem 5.1 that to evaluate $X(e^{-\pi\sqrt{n/3}})$, we need the value of L_n. For example, employing the value $L_1 = 1$ in Theorem 5.1, we get

$$X(e^{-\pi/\sqrt{3}}) = \left(\frac{\sqrt{3}-1}{3\sqrt{3}+3}\right)^{1/4}. \tag{5.2}$$

Theorem 5.2. For $q := e^{-2\pi\sqrt{n/3}}$, let

$$J_n = \frac{\phi(-q)}{3^{1/4}\phi(-q^3)},$$

then

$$X(e^{-4\pi\sqrt{n/3}}) = \left(\frac{1-\sqrt{3}J_n^2}{\sqrt{3}J_n^2+3}\right)^{1/2}.$$

Proof. We use the definition of J_n in Lemma 2.1 and simplify. ∎

We note that the parameter J_n is a particular case of the parameter $h'_{k,n}$, where $k = 3$.

Theorem 5.3. We have

$$\sqrt{3}\left(J_n^2 - \frac{1}{J_n^2}\right) + \left(\frac{L_n}{J_n}\right)^2 + \left(\frac{J_n}{L_n}\right)^2 = 0. \tag{5.3}$$

Proof. We replace q by $-q$ in Theorem 3.2 and use the definitions of L_n and J_n. ∎

Yi [9] evaluated explicit values of J_1 and J_3 in terms of parameter $h'_{k,n}$. More explicit values of J_n can be obtained by using explicit values of L_n in Theorem 5.3 and solving the resulting equations. For example, by setting $n = 1$, employing the value $L_1 = 1$ in Theorem 5.3, and solving the resulting equation for J_1, we get

$$J_1 = \left(\frac{3^{1/4}-1}{3^{1/4}+1}\right)^{1/4} = 2^{-1/4}\sqrt{\sqrt{3}-1}, \tag{5.4}$$

where the second expresion is due to Yi [9].

By Theorem 5.2 it is obvious that, to evaluate $X(e^{-4\pi\sqrt{n/3}})$ we need explicit values of J_n. For example, employing the value J_1 in Theorem 5.2, we get

$$X(e^{-4\pi/\sqrt{3}}) = \sqrt{\frac{2-3\sqrt{2}+\sqrt{6}}{6+3\sqrt{2}-6}}. \tag{5.5}$$

Theorem 5.4. For $q := e^{-\pi\sqrt{n/3}}$, let

$$I_n = \frac{\psi(q)}{3^{1/4}\psi(q^3)},$$

then

$$X(e^{-\pi\sqrt{n/3}}) = \frac{1}{3^{1/4}I_n}.$$

Proof. We use the definition of I_n in (1.5). ■

The parameter I_n is a particular case of the parameter $g'_{k,n}$, where $k = 3$ defined in (1.9). Baruah and Saikia [2] evaluated I_n for $n = 1, 2, 3, 4, 7, 9, 12, 16, 20, 36, 64$. Employing the values of I_n in Theorem 5.4 one can easily calculate explicit values of $X(e^{-\pi\sqrt{n/3}})$.

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