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# Existence of Three Solutions to a Neumann Problem for Elliptic Equations Driven by a $p$-Laplacian Operator

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**Abstract**

We investigate the existence of three weak solutions for a Neumann boundary value problem driven by a $p$-Laplacian operator. The technical approach is fully based on a three critical points theorem.

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## 1. Introduction

Throughout the paper, $\Omega \subset R^N (N \geq 1)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$ and $f : \Omega \times R \to R$ is a $L^1$- Carathéodory function.

**Remark 1.1.** We recall that a function $f : \Omega \times R \to R$ is said to be $L^1$-Carathéodory if $(\delta_1)$ $x \to f(x,t)$ is measurable for every $t \in R$; $(\delta_2)$ $t \to f(x,t)$ is continuous for almost every $x \in \Omega$; $(\delta_3)$ for every $\varrho > 0$ there exists a function $l_\varrho \in L^1(\Omega)$ such that

$$\sup_{|t| \leq \varrho} |f(x,t)| \leq l_\varrho(x)$$

for almost every $x \in \Omega$.

We are interested to study the following boundary value problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator, $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{\Omega} a > 0$ and $\lambda > 0$, based on a very recent three critical points theorem due to Bonanno and Marano [1].

In the sequel, X will denote the Sobolev space $W^{1,p}(\Omega)$ equipped with the norm

$$||u|| = \left(\int_\Omega (|\nabla u(x)|^p + a(x)|u(x)|^p)dx\right)^{1/p}.$$

Put

$$F(x,t) = \int_0^t f(x,\xi)d\xi$$

for each $(x,t) \in \Omega \times R$, and

$$c = \sup_{u \in X\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}} |u(x)|}{||u||}.$$

Since $p > N$, one has $c < +\infty$. In addition, if $\Omega$ is convex, it is known [2] that

$$\sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}} |u(x)|}{||u||} \leq 2^{\frac{p-1}{p}} \times \max\left\{\left(\frac{1}{||a||_1}\right)^{\frac{1}{p}}; \frac{\mathrm{diam}(\Omega)}{N^{\frac{1}{p}}}\left(\frac{p-1}{p-N}m(\Omega)\right)^{\frac{p-1}{p}} \frac{||a||_\infty}{||a||_1}\right\}$$

where $m(\Omega)$ is the Lebesgue measure of the set $\Omega$, and equality occurs when $\Omega$ is a ball.

By a solution (weak) of problem (1), we mean any $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega (|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)+a(x)|u(x)|^{p-2}u(x)v(x))dx - \lambda\int_\Omega f(x,u(x))v(x)dx = 0$$

for every $v \in W^{1,p}(\Omega)$.

For a thorough account on the subject we refer to [3-9] and therein references.

## 2. Main results

First we here recall for the reader's convenience Theorem 2.6 of [1] with $\Psi$ replaced by $-J$:

**Theorem 2.1.** Let X be a reflexive real Banach space, let $\Phi : X \longrightarrow R$ be a sequentially weakly lower semicontinuous, coercive and continuously $G\widehat{a}teaux$ differentiable whose $G\widehat{a}teaux$ derivative admits a continuous inverse on $X^*$, and let $J : X \longrightarrow R$ be a sequentially weakly lower semicontinuous and continuously $G\widehat{a}teaux$ differentiable functional whose $G\widehat{a}teaux$ derivative is compact. Assume that there exist $r \in R$ and $u_0, u_1 \in X$ with $\Phi(u_0) < r < \Phi(u_1)$ and $J(u_0) = 0$, such that

(i) $\displaystyle \sup_{u\in\Phi^{-1}(]-\infty,r])} (-J(u)) < (r - \Phi(u_0))\frac{-J(u_1)}{\Phi(u_1) - \Phi(u_0)}$,

(ii) for each $\lambda \in \Lambda_r := \left]\frac{\Phi(u_1) - \Phi(u_0)}{-J(u_1)}, \frac{r - \Phi(u_0)}{\sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u))}\right[$ the functional $\Phi + \lambda J$ is coercive.

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi + \lambda J$ has at least three distinct critical points in $X$.

Now we formulate our main result as follows:

**Theorem 2.2.** Let $f : \Omega \times R \to R$ be a $L^1$-Carath$\acute{e}$odory function, and denote $F(x, t) = \int_0^t f(x, \xi)d\xi$ for each $(x, t) \in \Omega \times R$. Assume that there exist a positive constant $r$ and a function $w \in X$ such that
$(\alpha_1)$ $||w||^p > pr$;
$(\alpha_2)$ $\dfrac{\int_\Omega \sup_{t\in[-c\sqrt[p]{pr},\, c\sqrt[p]{pr}]} F(x, t)dx}{r} < p\dfrac{\int_\Omega F(x, w(x))dx}{||w||^p}$;
$(\alpha_3)$ $\displaystyle \limsup_{|t|\to+\infty} \frac{F(x, t)}{|t|^p} < \frac{\int_\Omega \sup_{t\in[-c\sqrt[p]{pr},\, c\sqrt[p]{pr}]} F(x, t)dx}{m(\Omega)pc^p r}$ uniformly with respect to $x \in \Omega$.
Then, for each $\lambda \in \left[\dfrac{||w||^p}{p\int_\Omega F(x, w(x))dx}, \dfrac{r}{\int_\Omega \sup_{t\in[-c\sqrt[p]{pr},\, c\sqrt[p]{pr}]} F(x, t)dx}\right]$ the problem (2.1) admits at least three weak solutions in $X$.

*Proof.* In order to apply Theorem A, we begin by setting

$$\Phi(u) = \frac{1}{p}||u||^p \tag{2.1}$$

and

$$J(u) = -\int_\Omega F(x, u(x))dx \tag{2.2}$$

for each $u \in X$. Since $p > N$, $X$ is compactly embedded in $C^0(\overline{\Omega})$ and it is well known that $\Phi$ and $J$ are well defined and continuously $G\widehat{a}teaux$ differentiable functionals whose $G\widehat{a}teaux$ derivatives at the point $u \in X$ are the functionals $\Phi'(u),\ J'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega (|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) + a(x)|u(x)|^{p-2}u(x)v(x))dx$$

and

$$J'(u)(v) = -\int_\Omega f(x, u(x))v(x)dx$$

for every $v \in X$, respectively, as well as $J$ is sequentially weakly lower semicontinuous and $J' : X \to X^*$ is a compact operator.

We claim that $\Phi'$ admits a continuous inverse on $X^*$. To keep our claim, first we shall show that $\Phi'$ is a uniformly monotone operator in $X$. Moreover, taking into account (2.2) of [10], for every $u,\ v \in X$ there exists a positive constant $c_p$ such that

$$< |\nabla u(x)|^{p-2}\nabla u(x) - |\nabla v(x)|^{p-2}\nabla v(x), \nabla u(x) - \nabla v(x) >\geq c_p|\nabla u(x) - \nabla v(x)|^p$$

and

$$< |u(x)|^{p-2}u(x) - |v(x)|^{p-2}v(x), u(x) - v(x) >\geq c_p|u(x) - v(x)|^p$$

where $< .,\ . >$ denotes the usual inner product in $R^N$. So, we have

$$(\Phi'(u) - \Phi'(v))(u - v) \geq c_p||u - v||^p$$

for every $u, v \in X$, which means that $\Phi'$ is uniformly monotone. Therefore, since $\Phi$ is coercive and hemicontinuous in $X$, by applying Theorem 26.A. of [11], we have that $\Phi'$ admits a continuous inverse on $X^*$. Using again that $\Phi'$ is monotone, we obtain that $\Phi$ is sequentially weakly lower semi continuous (see [11, Proposition 25.20]).

Choose $u_0 = 0$ and $u_1 = w$, from $(\alpha_1)$ and (2) we get $\Phi(u_0) < r < \Phi(u_1)$, and by (3) we have $J(u_0) = 0$. Moreover, since

$$\sup_{x\in\Omega} |u(x)| \leq c||u|| \tag{2.3}$$

for each $u \in X$, we obtain

$$\begin{aligned}\Phi^{-1}(]-\infty, r]) &= \{u \in X; \Phi(u) \leq r\} \\ &= \left\{u \in X; ||u|| \leq \sqrt[p]{pr}\right\} \\ &\subseteq \left\{u \in X; |u(x)| \leq c\sqrt[p]{pr} \ \text{ for all } x \in \Omega\right\},\end{aligned}$$

and it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])} (-J(u)) &= \sup_{u\in\Phi^{-1}(]-\infty,r])} \int_\Omega F(x, u(x))dx \\ &\leq \int_\Omega \sup_{t\in[-c\sqrt[p]{pr},\ c\sqrt[p]{pr}]} F(x, t)dx.\end{aligned}$$

Therefore, owing to $(\alpha_2)$, we have

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u)) &= \sup_{u\in\Phi^{-1}(]-\infty,r])}\int_\Omega F(x,u(x))dx \\
&\leq \int_\Omega \sup_{t\in[-c\sqrt[p]{pr},\ c\sqrt[p]{pr}]} F(x,t)dx \\
&< pr\frac{\int_\Omega F(x,w(x))dx}{||w||^p} \\
&= (r-\Phi(u_0))\frac{-J(u_1)}{\Phi(u_1)-\Phi(u_0)},
\end{aligned}$$

namely, Assumption (i) of Theorem A is fulfilled. Furthermore from $(\alpha_3)$ there exist two constants $\gamma,\ \tau \in R$ with

$$0<\gamma<\frac{\int_\Omega \sup_{t\in[-c\sqrt[p]{pr},\ c\sqrt[p]{pr}]} F(x,t)dx}{r}$$

such that

$$pc^p m(\Omega)F(x,t)\leq \gamma|t|^p+\tau \ \text{ for a.e. } x\in\Omega.$$

Fix $u\in X$. Then

$$F(x,u(x))\leq \frac{1}{pc^p m(\Omega)}(\gamma|u(x)|^p+\tau) \ \text{ for a.e. } x\in\Omega. \tag{2.4}$$

So, for any fixed

$$\lambda\in \left]\frac{||w||^p}{p\int_\Omega F(x,w(x))dx},\frac{r}{\int_\Omega \sup_{t\in[-c\sqrt[p]{pr},\ c\sqrt[p]{pr}]} F(x,t)dx}\right[,$$

from (2), (3), (4) and (5), we have

$$\begin{aligned}
\Phi(u)+\lambda J(u) &= \frac{1}{p}||u||^p-\lambda\int_\Omega F(x,u(x))dx \\
&\geq \frac{1}{p}||u||^p-\frac{\lambda}{pc^p m(\Omega)}(\gamma\int_\Omega |u(x)|^p dx+m(\Omega)\tau) \\
&\geq \frac{1}{p}||u||^p-\frac{\lambda}{pc^p m(\Omega)}(\gamma c^p m(\Omega)||u||^p+m(\Omega)\tau) \\
&\geq \frac{1}{p}\left(1-\gamma\frac{r}{\int_\Omega \sup_{t\in[-c\sqrt[p]{pr},\ c\sqrt[p]{pr}]} F(x,t)dx}\right)||u||^p \\
&- \frac{r\tau}{pc^p\int_\Omega \sup_{t\in[-c\sqrt[p]{pr},\ c\sqrt[p]{pr}]} F(x,t)dx},
\end{aligned}$$

and thus

$$\lim_{||u||\to+\infty}(\Phi(u)+\lambda J(u))=+\infty,$$

which means the functional $\Phi+\lambda J$ is coercive. So, Assumption (ii) of Theorem A is satisfies. Now, we can apply Theorem A. Hence, by using Theorem A, taking into account that the weak solutions of (1) are exactly the solutions of the equation $\Phi'(u)+\lambda J'(u)=0$, the problem (1) admits at least three weak solutions. ■

**Remark 2.3.** Note $F(x,0)=0$, so $\int_\Omega \sup_{t\in[-c\sqrt[p]{pr},\, c\sqrt[p]{pr}]} F(x,t)dx \geq 0$ and now by applying $(\alpha_2)$ since $||w||^p>0$, we have that $\int_a^b F(x,w(x))dx>0$.

Let us here give a consequence of Theorem 2.2 for a fixed test function $w$.

**Corollary 2.4.** Let $f:\Omega\times R\to R$ be a $L^1$-Carathéodory function, and denote $F(x,t)=\int_0^t f(x,\xi)d\xi$ for each $(x,t)\in\Omega\times R$. Assume that there exist two positive constants $\theta$ and $\tau$ with $\frac{\theta}{c}<\tau$ such that

$(\alpha_4)$ $\dfrac{\int_\Omega \sup_{t\in[-\theta,\theta]}F(x,t)dx}{\theta^p}<\dfrac{\int_\Omega F(x,\tau||a||_1^{-\frac{1}{p}})dx}{(c\tau)^p}$;

$(\alpha_5)$ $\limsup_{|t|\to+\infty}\dfrac{F(x,t)}{|t|^p}<\dfrac{\int_\Omega \sup_{t\in[-\theta,\theta]}F(x,t)dx}{m(\Omega)\theta^p}$ uniformly with respect to $x\in\Omega$.

Then, for each $\lambda$ satisfying $\frac{\lambda}{p}\in\left]\dfrac{\tau^p}{\int_\Omega F(x,\tau||a||_1^{-\frac{1}{p}})dx},\dfrac{\theta^p}{c^p\int_\Omega \sup_{t\in[-\theta,\theta]}F(x,t)dx}\right[$ the problem (1) admits at least three weak solutions in $X$.

*Proof.* We claim that all the assumptions of Theorem 2.2 are satisfied by choosing

$$w(x)=\tau||a||_1^{-\frac{1}{p}} \tag{2.5}$$

and $r=\frac{1}{p}(\frac{\theta}{c})^p$. It follows from (2.5) that $w\in X$ and $||w||^p=\tau^p$, so the assumption $\tau>\frac{\theta}{c}$ follows that Assumption $(\alpha_1)$ is fulfilled. Also, from $(\alpha_4)$ and $(\alpha_5)$ we arrive at $(\alpha_2)$ and $(\alpha_3)$, respectively. Hence, Theorem 2.2 follows the conclusion. ■

We now present a particular case of Corollary 2.4, in which the function $f$ has separated variables.

**Corollary 2.5.** Let $\tilde{f}_1\in L^1(\Omega)$ be a positive function and $\tilde{f}_2\in C(R)$ be a function. Put $\tilde{F}(t)=\int_0^t \tilde{f}_2(\xi)d\xi$ for all $t\in R$. Assume that there exist two positive constants $\theta$

and $\tau$ with $\frac{\theta}{c} < \tau$ such that

$(\alpha_6)$ $\frac{\max_{t\in[-\theta,\theta]} \tilde{F}(t)}{\theta^p} < \frac{\tilde{F}(\tau||a||_1^{-\frac{1}{p}})}{(c\tau)^p}$;

$(\alpha_7)$ $\frac{\tilde{f}_1(x)}{\int_\Omega \tilde{f}_1(x)dx} \limsup_{|t|\to+\infty} \frac{\tilde{F}(t)}{|t|^p} < \frac{\max_{t\in[-\theta,\theta]} \tilde{F}(t)}{m(\Omega)\theta^p}$ uniformly with respect to $x \in \Omega$.

Then, for each $\lambda$ satisfying $\frac{\lambda}{p\int_\Omega \tilde{f}_1(x)dx} \in \left]\frac{\tau^p}{\tilde{F}(\tau||a||_1^{-\frac{1}{p}})}, \frac{\theta^p}{c^p \max_{t\in[-\theta,\theta]} \tilde{F}(t)}\right[$ the problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda \tilde{f}_1(x)\tilde{f}_2(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least three weak solutions in $X$.

*Proof.* Set $f(x,t) = \tilde{f}_1(x)\tilde{f}_2(t)$ for each $(x,u) \in \Omega \times R$. Since

$$F(x,t) = \widetilde{f_1}(x)\widetilde{F}(t),$$

from $(\alpha_6)$ and $(\alpha_7)$ we obtain $(\alpha_4)$ and $(\alpha_5)$, respectively. Hence, Corollary 2.4 yields the conclusion. ■

Let us present an application of Corollary 2.5.

**Example 2.6.** Consider the problem

$$\begin{cases} -\Delta_3 u + \frac{x^2+y^2}{\pi}|u|u = \lambda(2e^{-u^2}u^9(5-u^2)+1) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

where $\Omega = \{(x,y) \in R^2; x^2+y^2 < 9\}$. We choose $p = 3$, $a(x,y) = \frac{x^2+y^2}{\pi}$ for each $(x,y) \in \Omega$, $\widetilde{f_1}(x,y) = 1$ for each $(x,y) \in \Omega$ and $\widetilde{f_2}(t) = 2e^{-t^2}t^9(5-t^2)+1$ for each $t \in R$. Note that $\widetilde{F}(t) = e^{-t^2}t^{10} + t$ for all $t \in R$, by choosing $\theta = 1$ and $\tau = 3$, taking into account that $c = \left(\frac{1536}{\pi}\right)^{\frac{1}{3}}$, it is easy to check that all hypotheses of Corollary 2.5 are satisfied. Hence, Corollary 2.5 is applicable to the problem (2.6) for each $\lambda$ satisfying

$$\frac{\lambda}{27\pi} \in \left]\frac{27}{3^{10}\left(\frac{81}{2}\right)^{-\frac{10}{3}} e^{-9\left(\frac{81}{2}\right)^{-\frac{2}{3}}} + 3\left(\frac{81}{2}\right)^{-\frac{1}{3}}}, \frac{\pi}{1536(e^{-1}+1)}\right[.$$

Let us here give the following special case of Corollary 2.5 when $N = 1$ and $p = 2$. For simplicity, we fix $\Omega = (\beta_1, \beta_2)$. In this case, we have

$$c = \sqrt{2}\max\{||a||_1^{-\frac{1}{2}}, (\beta_2 - \beta_1)^{\frac{3}{2}}||a||_\infty||a||_1^{-1}\}.$$

**Corollary 2.7.** Let $\tilde{f_1} \in L^1(\beta_1, \beta_2)$ be a positive function and $\tilde{f_2} \in C(R)$ be a function such that $\tilde{f_2}(t) \geq 0$ for all $t \in [-\theta, \theta]$. Put $\tilde{F}(t) = \int_0^t \tilde{f_2}(\xi)d\xi$ for all $t \in R$, there exist two positive constants $\theta$ and $\tau$ with $\frac{\theta}{c} < \tau$ such that

$(\alpha_8)$ $\dfrac{\tilde{F}(\theta)}{\theta^2} < \dfrac{\tilde{F}(\tau||a||_1^{-\frac{1}{2}})}{(c\tau)^2}$;

$(\alpha_9)$ $\dfrac{\tilde{f_1}(x)dx}{\int_{\beta_1}^{\beta_2}\tilde{f_1}(x)dx}\limsup_{|t|\to+\infty}\dfrac{\tilde{F}(t)}{|t|^2} < \dfrac{\tilde{F}(\theta)}{(\beta_2-\beta_1)\theta^2}$.

Then, for each $\lambda$ satisfying $\dfrac{\lambda}{2\int_{\beta_1}^{\beta_2}\tilde{f_1}(x)dx} \in \left[\dfrac{\tau^2}{\tilde{F}(\tau||a||_1^{-\frac{1}{2}})}, \dfrac{\theta^2}{c^2\tilde{F}(\theta)}\right]$ the problem

$$\begin{cases} -u'' + a(x)|u|u = \lambda\tilde{f_1}(x)\tilde{f_2}(u) & \text{in } (\beta_1, \beta_2), \\ u'(\beta_1) = u'(\beta_2) = 0 \end{cases}$$

admits at least three weak solutions in $W^{1,2}(\beta_1, \beta_2)$.

We conclude this paper by giving a simple consequence of Corollary 2.7 when $a(x) \equiv 1$ for all $x \in (0, 1)$.

**Corollary 2.8.** Let $\tilde{f_2} \in C(R)$ be a function such that $\tilde{f_2}(t) \geq 0$ for all $t \in [-\theta, \theta]$. Put $\tilde{F}(t) = \int_0^t \tilde{f_2}(\xi)d\xi$ for all $t \in R$, there exist two positive constants $\theta$ and $\tau$ with $\theta < \sqrt{2}\tau$ such that

$(\alpha_{10})$ $\dfrac{\tilde{F}(\theta)}{\theta^2} < \dfrac{1}{2}\dfrac{\tilde{F}(\tau)}{\tau^2}$;

$(\alpha_{11})$ $\limsup_{|t|\to+\infty}\dfrac{\tilde{F}(t)}{|t|^2} < \dfrac{\tilde{F}(\theta)}{\theta^2}$.

Then, for each $\lambda$ satisfying $\dfrac{\lambda}{2} \in \left]\dfrac{\tau^2}{\tilde{F}(\tau)}, \dfrac{\theta^2}{2\tilde{F}(\theta)}\right[$ the problem

$$\begin{cases} -u'' + |u|u = \lambda\tilde{f_2}(u) & \text{in } (0, 1), \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least three weak solutions in $W^{1,2}(0, 1)$.

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