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Two New Thinning Operators and their Applications

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Abstract

Binomial thinning operator was introduced to define α-monotone distributions on the lattice of integers. This operator is actually a compound of Bernoulli i.i.d. random variables which was also used to produce self-decomposable distributions on the lattice of nonnegative integers. Several other different thinning operators have been defined to model discrete data, specially, in integer-valued time series. In this paper, we shall introduce two new thinning operators based on zero-inflated and inflated-parameter Bernoulli random variables. Then, based on these, we shall define two new concepts, namely, modified and generalized α-monotonicity and discuss their similar properties and characterizations. We shall also use our new operations to extend the notion of discrete self-decomposability.

AMS Subject Classification:
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1. Introduction

Thinning operators are probabilistic operations that can be applied to random counts. The basic idea is that count represents the random size of an imaginary population, and

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the thinning operation randomly deletes some of the members of this population. The first and most popular thinning operation is the binomial thinning operator. Steutel and van Harn (1979) defined binomial thinning operator (multiplication) as follows:

Definition 1.1. Let Y be an arbitrary nonnegative integer-valued random variable (r.v.) and $X_1, X_2, \ldots$ be i.i.d. Bernoulli r.v.'s with parameter $u \in [0, 1]$ (Ber(u)). Then

$$u \otimes Y = \sum_{i=1}^{Y} X_i \tag{1.1}$$

is called the binomial thinning operator(or u-fraction) of Y. Obviously, in terms of probability generating functions (pgf's) we have

$$P_{u\otimes Y}(t) = P_Y(1 - u + ut)$$

For the interpretation of the binomial thinning operation, consider a population of size Y at a certain time t. If we observe the same population at time $t + 1$, then the population may be shrunk, because some of the individuals have died between times t and $t + 1$. If the individuals die independently of each other, and if the probability of dying in between t and $t + 1$ is equal to $1 - u$ for all individuals, then the number of survivors is given by $u \otimes Y$. Steutel and van Harn (1979) used binomial thinning operator to define a discrete analogue of self-decomposability. They called a nonnegative discrete r.v. X self-decomposable, if for any $\alpha \in (0, 1)$ there exists a r.v. X_α independent of X such that $X =^d \alpha \otimes X + X_\alpha$ ($=^d$ means equality in distribution).

Steutel (1988) also defined a discrete analogue of α-monotonicity of Olshen and Savage (1970) in the term of binomial thinning operator as follows.

Definition 1.2. A nonnegative integer-valued r.v. X is α-monotone, if

$$X =^d U^{1/\alpha} \otimes Y, \tag{1.2}$$

where Y is a nonnegative integer-valued r.v. and U is a uniform distributed r.v. on (0, 1) ($U \sim U(0, 1)$) independent of Y.

He showed that a nonnegative integer-valued r.v. X with distribution $\{p_n\}_0^\infty$ and probability generating function P_X, is α-monotone if, and only if (iff)

$$P_X(t) = \alpha(1 - t)^{-\alpha} \int_t^1 (1 - w)^{\alpha-1} Q(w)dw, \tag{1.3}$$

where $Q(.)$ is the pgf of a nonnegative discrete r.v.. Or, equivalently, iff

$$(\alpha + n)p_n \geq (1 + n)p_{n+1}, \qquad \forall n \geq 0\,.$$

According to Steutel (1988), a discrete distribution $\{p_n\}_{-\infty}^\infty$ is called α-unimodal (about zero) if for some $\alpha > 0$,

$$\begin{cases} (\alpha - n)p_n \geq (1 - n)p_{n-1}, & n \leq 0 \\ (\alpha + n)p_n \geq (1 + n)p_{n+1}, & n \geq 0. \end{cases}$$

Obviously, 1-unimodality is just the usual unimodality and an α-unimodal (monotone) distribution is β-unimodal (monotone), provided that $\beta \geq \alpha$.

Alamatsaz (1993) considered the convolution property of α-monotone distributions. He proved that the convolution of a discrete α-monotone and a discrete β-monotone distribution is a discrete $(\alpha + \beta)$-monotone distribution. Wu and Dharmadhikari (1999) used this result to show a similar result for the convolution of α-unimodal distributions.

Alzaid and Al-Osh (1990) gave a similar characterization to that of Olshen and Savage (1970) for discrete α-monotone distributions. They proved that a nonnegative integer-valued r.v. X is α-monotone iff for every nonnegative bounded measurable function g, $t^{\alpha} E[g(t \otimes X)]$ is non-decreasing in $t \in (0, 1]$.

Binomial thinning operator has also been applied in data analysis. Mckenzie (1985) first applied binomial thinning operator for the analysis of integer-valued time series. He defined integer-valued autoregressive time series of the first order, denoted by INAR (1.1), as the recursion:

$$X_t = \alpha \otimes X_{t-1} + e_t$$

where e_t's are i.i.d. r.v.'s with range $N_0 = \{0, 1, \ldots\}$ and $\alpha \in [0, 1]$. This recursion can be used to model ordered count data. Let the probability of mortality in an imaginary population of size X_t at time t be $1-\alpha$. Then, clearly, X_t is a contribution of $\alpha \otimes X_{t-1}$ as the number of survivors after time $t-1$ and immigration of size (e_t) at time t. Also, the equality of the expectations of αX_t and $\alpha \otimes X_t$ was a motivation to replace multiplication by binomial thinning operator in autoregressive moving average (ARMA) time series to obtain integer-valued ARMA models in the literature. For more details see, e.g., Al-Osh and Alzaid (1987) and Weiß (2008a).

It has been pointed out that these models are useful, especially for processes of Poisson counts, but may lead to difficulties in the case of different count distributions. Therefore, several alternative thinning operators are successfully applied to define integer-valued ARMA models. For example, Latour (1998) defined generalized thinning operator

$$\alpha \otimes_{\beta} X = \sum_{i=1}^{X} X_i$$

where X_i's are i.i.d r.v.'s with mean α and variance β. See Weiß (2008b) for more examples in this regard.

In this article, we define two generalizations of binomial thinning operator by replacing Bernoulli i.i.d r.v.'s by zero-inflated Bernoulli and inflated parameter Bernoulli i.i.d r.v.'s.. Then, we shall extend the notions of discrete α-monotonicity, α-unimodality and self-decomposability based on these generalized thinning operators. Some characterizations will be provided for some inflated parameter generalized power series distributions.

Nearly all discrete distributions that are widely used for counting, in practice, are members of the family of power series distributions. This family has the probability mass function (pmf) given by

$$p_k = \frac{a(k)}{g(\theta)}\theta^k, \qquad k \in N_0, \tag{1.4}$$

where N_0 is the set of nonnegative integers, $\theta > 0$ is the series parameter, $a(.)$ is a nonnegative function on N_0 and $g(\theta) = \sum_{k \in N_0} a(k)\theta^k$ is the normalizing constant. Clearly, pgf of (1.4) is given by $P(t) = \frac{g(\theta t)}{g(\theta)}$, $t \in [0, 1]$. Binomial, negative binomial, Poisson and logarithmic series distributions are well-known examples of this family.

Let's first recall inflated-parameter generalized power series distributions. For details we refer to Kolev et al. (2000) and Minkova (2002). Let ξ be an arbitrary nonnegative integer-valued r.v. such that $P(\xi = j) = p_j$, $j = 0, 1, 2, \ldots$ and $\Sigma_{j=0}^{\infty} p_j = 1$, and $P_\xi(t) = E(t^\xi)$ be its pgf. If an extra proportion of zeros, $\rho \in [0, 1]$, is added to zeros of the r.v. ξ, while decreasing the remaining proportions in an appropriate way, the zero-inflated modification η of ξ is given by

$$P(\eta = j) = \begin{cases} \rho + (1-\rho)p_0, & j = 0 \\ (1-\rho)p_j, & j = 1, 2, \ldots. \end{cases}$$

Clearly, the pgf of η is $P_\eta(t) = \rho + (1-\rho)P_\xi(t)$. Thus, if $\rho = 1$, the corresponding zero-inflated distribution degenerates at zero and if $\rho = 0$, there is no inflation, i.e., $P_\eta(t) = P_\xi(t)$.

For most cases, the inflation parameter ρ lies between 0 and 1, although it may also take negative values, provided that $P(\eta = 0) > 0$, i.e., $\rho \geq -\frac{p_0}{1-p_0}$. This latter case corresponds to the opposite phenomena, i.e., excludes a proportion of zeros from the basic discrete distribution. In this paper, we consider the case where $\rho \in [0, 1]$. For example, the distribution of a zero-inflated Bernoulli r.v. X with parameter $\pi \in [0, 1]$ and the inflation parameter ρ, denoted by $X \sim$ZIBer(π, ρ), is given by

$$P(X = j) = \begin{cases} \rho + (1-\rho)(1-\pi), & j = 0 \\ (1-\rho)\pi, & j = 1. \end{cases} \tag{1.5}$$

Let $\{W_1, W_2, \ldots\}$ be a sequence of independent ZIBer(π, ρ) r.v.'s, and V be the number of trials we need to achieve the first observed success in the sequence. Then, obviously, pmf of V is given by

$$P(V = k) = [(1-\pi)(1-\rho) + \rho]^{k-1}(1-\rho)\pi, \qquad k = 1, 2, \ldots.$$

Now, define a r.v. S by the following relation

$$P(S = k) = \begin{cases} \pi, & k = 0 \\ (1-\pi)[(1-\pi)(1-\rho) + \rho]^{k-1}(1-\rho)\pi, & k = 1, 2, \ldots. \end{cases}$$

Then, S is called an inflated-parameter geometric r.v., and is denoted by $S \sim$ IGe(π, ρ). The pgf of S is given by

$$P_S(t) = \frac{\pi(1 - t\rho)}{1 - t[1 - \pi + \rho\pi]},$$

which is actually the pgf of $S = X_1 + X_2 + \cdots + X_N$, where N is a geometric r.v. with parameter $\pi \in (0, 1)$ and pgf $P_N(t) = \dfrac{\pi}{1 - (1 - \pi)t}$, $t \in [0, 1]$ $(Ge_0(\pi))$, and $X_1, X_2, \ldots$ are geometric r.v.'s with parameter $1-\rho$ and pgf $P_X(t) = \dfrac{t(1-\rho)}{1 - t\rho}$, $t \in [0, 1]$ $(Ge_1(1 - \rho))$, independent of r.v. N. If N is a r.v. distributed according to the binomial, negative binomial or Poisson distributions, then $S = X_1 + X_2 + \cdots + X_N$ is inflated-parameter binomial, negative binomial or Poisson r.v., respectively. Generally, if N is a power series distributed r.v. with pmf (1.4), then the pgf of S is of the form

$$P_S(t) = \frac{g(\theta P_X(t))}{g(\theta)}, \qquad t \in [0, 1].$$

By Taylor's expansion of the pgf, the pmf of S is given by

$$P(S = k) = \sum_{i=1}^{k} \binom{k-1}{i-1} (1 - \rho)^i \rho^{k-i} \frac{a(i)}{g(\theta)} \theta^i .$$

For example, the pmf of an inflated-parameter Bernoulli r.v. X with parameter $\pi \in [0, 1]$ and the inflation parameter ρ, denoted by $X \sim$IBer(π, ρ), is given by

$$P(X = j) = \begin{cases} 1 - \pi, & j = 0 \\ \pi\rho^{k-1}(1 - \rho), & j = 1, 2, \ldots . \end{cases} \tag{1.6}$$

It is known that if $X_1, X_2, \ldots, X_n$ are i.i.d IBer(π, ρ) r.v.'s, then $T_n = \sum_{i=1}^{n} X_i$ is an inflated-parameter binomial r.v. with parameters n, π and ρ, denoted by IBin(n, π, ρ), with pmf

$$P(T_n = k) = \sum_{i=1}^{min(k,n)} \binom{n}{i} \binom{k-1}{i-1} [\pi(1 - \rho)]^i (1 - \pi)^{n-i} \rho^{k-i} . \tag{1.7}$$

Similarly, if $X_1, X_2, \ldots, X_n$ are i.i.d IGe(π, ρ) r.v.'s, then $T_n = \sum_{i=1}^{n} X_i$ is an inflated-parameter negative binomial r.v. with parameters n, π and ρ, denoted by INB(n, π, ρ). Also, an inflated-parameter Poisson distribution with parameter $\lambda > 0$ and the inflation parameter ρ, denoted by IPo(λ, ρ), is the weak limit of IBin(n, π, ρ) as $\pi \to 0$ and $n \to \infty$ such that $\lambda = n\pi$ remains constant (see Kolev et al., 2000, for a proof). ■

In section 2, we shall introduce a modification of the binomial operator $\otimes$ using zero-inflated Bernoulli r.v.'s and discuss their properties. Section 3 concerns with a generalization of the concept using inflated-parameter Bernoulli r.v.'s. Finally in section 4, using the new notions, we shall extend the concept of discrete self-decomposability of Steutel and van Harn (1979).

2. ρ-Modified α-Monotone Discrete Distributions

We shall first consider an extension of the binomial thinning operator $\otimes$ introduced by Steutel and van Harn (1979), based on zero-inflated Bernoulli r.v.'s. Then, accordingly, we introduce a modification of α-monotonicity.

Definition 2.1. Let Y be a discrete r.v. on N_0 and $X_1, X_2, \ldots$ be i.i.d. ZIBer(u, ρ) r.v.'s with $\rho, u \in [0, 1]$. Then, we call the operator $\otimes^\rho$ defined by

$$u \otimes^\rho Y = \sum_{i=1}^{Y} X_i \tag{2.1}$$

the zero-inflated thinning operator with inflation parameter ρ.

In the case $\rho = 0$ (no inflation), zero-inflated thinning operator (2.1) is just the (usual) binomial thinning operator. Further, we can see that by

$$u \otimes^\rho Y =^d u(1-\rho) \otimes Y \tag{2.2}$$

because, by (1.5), the pgf of $u \otimes^\rho Y$ is

$$\begin{aligned} P_{u\otimes^\rho Y}(t) &= EE(t^{u\otimes^\rho Y}|Y) \\ &= EE(t^{\sum_{i=1}^{Y} X_i}|Y) \\ &= E[E(t^{X_1})]^Y \\ &= E[\rho + (1-\rho)(1-u+ut)]^Y \\ &= P_Y(1-u(1-\rho)+u(1-\rho)t) \\ &= P_{\upsilon\otimes Y}(t)\,, \end{aligned}$$

where $\upsilon = u(1-\rho)$.

Definition 2.2. A r.v. X on N_0 is called ρ-modified α-monotone((ρ, α)-MM) if it can be represented as

$$X =^d U^{1/\alpha} \otimes^\rho Y, \tag{2.3}$$

where Y is a r.v. defined on N_0 independent of $U \sim U(0, 1)$ and $\rho \in [0, 1]$.

Obviously, an $(0, \alpha)$-MM r.v. X is an α-monotone r.v. in the usual sense.

Theorem 2.3. A r.v. X with distribution $\{p_n\}_0^\infty$ is (ρ, α)-MM iff $X =^d \sum_{i=1}^{N} Z_i$, where $Z_1, Z_2, \ldots$ are i.i.d. Ber$(1-\rho)$ r.v.'s and N is a nonnegative integer-valued α-monotone r.v. independent of Z_i's.

Proof. Since if X satisfies (2.3) then, by (2.2), we readily obtain

$$\begin{aligned} X &=^d U^{1/\alpha} \otimes^\rho Y \\ &=^d ((1-\rho)U^{1/\alpha}) \otimes Y \\ &=^d (1-\rho) \otimes (U^{1/\alpha} \otimes Y) \\ &=^d (1-\rho) \otimes N, \end{aligned}$$

where $N =^d U^{1/\alpha} \otimes Y$ is an α-monotone r.v. and vice versa, as required. ∎

Corollary 2.4. A r.v. X on N_0 is (ρ, α)-MM iff its pgf can be written as

$$P_X(t) = \alpha[(1-\rho)(1-t)]^{-\alpha} \int_{1-(1-\rho)(1-t)}^{1} (1-w)^{\alpha-1} Q(w)dw,$$

where $Q(.)$ is the pgf of a r.v. on N_0.

Proof. Using Theorem 2.3, we can write

$$\begin{aligned} P_X(t) &= E(t^{(1-\rho)\otimes(U^{1/\alpha}\otimes Y)}) \\ &= P_{U^{1/\alpha}\otimes Y}(1-(1-\rho)(1-t)). \end{aligned}$$

Thus, by (1.3) and the uniqueness property of pgf's, we have the result. ∎

Also, it is easy to show that

Corollary 2.5. A (ρ, α)-MM r.v. is α-monotone, but not necessarily vice versa.

Remark 2.6. By Theorem 2.3, it is easy to see that (ρ, α)-MM distribution is (ρ, β)-MM, provided that $\beta \geq \alpha$.

Now, we prove the following convolution property of ρ-modified α-monotone distributions similar that of Alamatsaz (1993).

Theorem 2.7. Convolution of a (ρ, α)-MM and a (ρ, β)-MM distribution is a $(\rho, \alpha+\beta)$-MM distribution.

Proof. Let X and Y be independent and (ρ, α)-MM and (ρ, β)-MM r.v.'s, respectively.

Then, the pgf of the convolution is given by

$$\begin{aligned}P(t) &= P_{X+Y}(t)\\ &= P_X(t).P_Y(t)\\ &= \alpha[(1-\rho)(1-t)]^{-\alpha}\int_{1-(1-\rho)(1-t)}^{1}(1-w)^{\alpha-1}Q_1(w)dw.\quad(\text{by Corollary 2.1})\\ &\quad .\beta[(1-\rho)(1-t)]^{-\beta}\int_{1-(1-\rho)(1-t)}^{1}(1-w)^{\beta-1}Q_2(w)dw\\ &= \alpha\beta[(1-\rho)(1-t)]^{-(\alpha+\beta)}Q(t), \end{aligned}\tag{2.4}$$

where Q_1 and Q_2 are some pgf's and

$$Q(t) = \int_{1-(1-\rho)(1-t)}^{1}(1-w)^{\alpha-1}Q_1(w)dw.\int_{1-(1-\rho)(1-t)}^{1}(1-w)^{\beta-1}Q_2(w)dw.$$

Thus,

$$\begin{aligned}\frac{d}{dt}Q(t) &= -(1-\rho)[(1-\rho)(1-t)]^{\alpha+\beta-1}.\\ &\quad \Big\{Q_1(t)[(1-\rho)(1-t)]^{-\beta}\int_{1-(1-\rho)(1-t)}^{1}(1-w)^{\beta-1}Q_2(w)dw\\ &\quad +Q_2(t)[(1-\rho)(1-t)]^{-\alpha}\int_{1-(1-\rho)(1-t)}^{1}(1-w)^{\alpha-1}Q_1(w)dw\Big\}\\ &= -(1-\rho)[(1-\rho)(1-t)]^{\alpha+\beta-1}\left\{\frac{1}{\beta}Q_1(t)Q_2^*(t)+\frac{1}{\alpha}Q_2(t)Q_1^*(t)\right\}\\ &= -(1-\rho)[(1-\rho)(1-t)]^{\alpha+\beta-1}\frac{\alpha+\beta}{\alpha\beta}Q^*(t)\end{aligned}$$

where

$$Q_1^*(t) = \alpha[(1-\rho)(1-t)]^{-\alpha}\int_{1-(1-\rho)(1-t)}^{1}(1-w)^{\alpha-1}Q_1(w)dw,$$

$$Q_2^*(t) = \beta[(1-\rho)(1-t)]^{-\beta}\int_{1-(1-\rho)(1-t)}^{1}(1-w)^{\beta-1}Q_2(w)dw$$

and

$$Q^*(t) = \frac{\alpha}{\alpha+\beta}Q_1(t)Q_2^*(t) + \frac{\beta}{\alpha+\beta}Q_2(t)Q_1^*(t)$$

are some pgf's on N_0. Now, by integrating on $(1-(1-\rho)(1-t), 1)$ from both sides with respect to t, we have

$$Q(t) = \frac{\alpha+\beta}{\alpha\beta}\int_{1-(1-\rho)(1-t)}^{1}(1-w)^{\alpha+\beta-1}Q^*(w)dw.$$

Inserting this into (2.4), we get the result by Corollary 2.4. ■

Example 2.8. All α-monotone-compounding of i.i.d. Ber$(1-\rho)$ r.v.'s such as geometric distribution are (ρ, α)-MM ($\alpha \geq 1$), because $\mathrm{Ge}_0(\pi)$ is a monotone distribution and a $\mathrm{Ge}_0(\pi)$-compounding of i.i.d. Ber$(1-\rho)$ r.v.'s has been shown to be a $\mathrm{Ge}_0(p)$ distributed r.v., where $p = \dfrac{\pi}{1-(1-\pi)\rho}$ is a real value in (0, 1). Thus, by Theorem 2.3 and Remark 2.6, $\mathrm{Ge}_0(p)$ is a (ρ, α)-MM distribution, for all $p \in (0, 1)$ and $\alpha \geq 1$.

3. ρ-Generalized α-Monotone Discrete Distributions

Now, we extend binomial thinning operator $\otimes$ based on inflated-parameter Bernoulli r.v.'s.

Definition 3.1. Let Y be a discrete r.v. on N_0 and $X_1, X_2, \ldots$ be i.i.d. IBer(u, ρ) r.v.'s with $\rho, u \in [0, 1]$. Then, we call the operator $\otimes_\rho$ defined by

$$u \otimes_\rho Y = \sum_{i=1}^{Y} X_i \tag{3.1}$$

an inflated binomial thinning operator with inflation parameter ρ.

In the case $\rho = 0$ (no inflation), the inflated binomial thinning operator (3.1) is again the (usual) thinning operator $\otimes$ in (1.1).

We may also note that for the pgf of $u \otimes_\rho Y$ we have

$$\begin{aligned} P_{u\otimes_\rho Y}(t) &= EE(t^{u\otimes_\rho Y}|Y) \\ &= EE(t^{\sum_{i=1}^{Y} X_i}|Y) \\ &= E[E(t^{X_1})]^Y \\ &= E\left[1-\frac{u(1-t)}{1-t\rho}\right] && \text{(by (1.6))} \\ &= P_Y\left(1-u+u\frac{t(1-\rho)}{1-t\rho}\right) \\ &= P_{u\otimes Y}\left(\frac{t(1-\rho)}{1-t\rho}\right). && (3.2) \end{aligned}$$

Or, equivalently,

$$u \otimes_\rho Y =^d \sum_{i=1}^{u\otimes Y} Z_i, \tag{3.3}$$

where $Z_1, Z_2, \ldots$ are i.i.d. $\mathrm{Ge}_1(1-\rho)$ r.v.'s with pgf $P_Z(t) = \dfrac{t(1-\rho)}{1-t\rho}$, i.e., $u \otimes_\rho Y$ is a $u \otimes Y$-compounding of $\mathrm{Ge}_1(1-\rho)$ r.v.'s.

Remark 3.2. In the binomial thinning operator, we have $0 \otimes Y =^d 0$ and $1 \otimes Y =^d Y$, but in the inflated binomial case, we have

(a) $0 \otimes_\rho Y =^d 0$,

(b) $1 \otimes_\rho Y =^d \sum_{i=1}^{Y} Z_i$, where $Z_1, Z_2, \ldots$ are i.i.d. $\mathrm{Ge}_1(1-\rho)$ r.v.'s.

Theorem 3.3. Let Y be a r.v. defined on N_0 with pgf P_Y and u, v and ρ be real values in $[0, 1]$. Then, we have (a) $v \otimes (u \otimes_\rho Y) =^d w \otimes_{\rho_0} Y$, where $w = \dfrac{uv}{1-\rho(1-v)}$ and $\rho_0 = \dfrac{v\rho}{1-\rho(1-v)}$ are real values in $[0, 1]$. (b) $v \otimes_\rho (u \otimes Y) =^d uv \otimes_\rho Y =^d u \otimes_\rho (v \otimes Y)$.

Proof. (a) Since by (3.1) the pgf of $u \otimes_\rho Y$ is $P_Y\left(1 - \dfrac{u(1-t)}{1-t\rho}\right)$, the pgf of $v \otimes (u \otimes_\rho Y)$ is given by

$$\begin{aligned}
P_{v\otimes(u\otimes_\rho Y)}(t) &= P_Y\left(1 - \frac{u(1-(1-v+vt))}{1-\rho(1-v+vt)}\right) \\
&= P_Y\left(1 - \frac{uv(1-t)}{1-\rho(1-v)-\rho vt}\right) \\
&= P\left(1 - \frac{w(1-t)}{1-\rho_0 t}\right) \\
&= P_{w\otimes_{\rho_0} Y}(t)\,.
\end{aligned}$$

Therefore, $v \otimes (u \otimes_\rho Y) =^d w \otimes_{\rho_0} Y$, where $w = \dfrac{uv}{1-\rho(1-v)}$ and $\rho_0 = \dfrac{v\rho}{1-\rho(1-v)}$ are real values in $[0, 1]$.
(b) Using a similar argument as in part (a) and noting that $P_Y(1-u+ut)$ is the pgf of $u \otimes Y$, we now have

$$\begin{aligned}
P_{v\otimes_\rho(u\otimes Y)}(t) &= P_Y\left(1 - u + u\left(1 - \frac{v(1-t)}{1-\rho t}\right)\right) \\
&= P_Y\left(1 - \frac{uv(1-t)}{1-\rho t}\right) \\
&= P_{uv\otimes_\rho Y}(t)\,.
\end{aligned}$$

Therefore, $v \otimes_\rho (u \otimes Y) =^d uv \otimes_\rho Y$. The last part is valid by symmetry. ■

Corollary 3.4.

(a) $v \otimes (u \otimes_\rho Y) =^d u \otimes_\rho Y$ iff $v = 1$,

(b) $u \otimes (v \otimes_\rho Y) =^d v \otimes (u \otimes_\rho Y)$ iff $u = v$, and

(c) $v \otimes_\rho (u \otimes Y) =^d v \otimes (u \otimes_\rho Y)$ iff $v = 1$.

We can extend the result of Theorem 3.3 in the following manner.

Theorem 3.5. Under the assumptions of Theorem 3.3, $v \otimes_{\rho'} (u \otimes_\rho Y) =^d w \otimes_{\rho_0} Y$, where $w = \dfrac{uv}{1-\rho(1-v)}$ and $\rho_0 = \dfrac{\rho'(1-\rho)+v\rho}{1-\rho(1-v)}$ are real values in [0, 1].

Proof. By a similar argument used in Theorem 3.3, we obtain the result. ■

Now, we can define ρ-generalized α-monotone discrete distributions.

Definition 3.6. We call a r.v. X on N_0 ρ-generalized α-monotone((ρ, α)-GM) if it can be represented as

$$X =^d U^{1/\alpha} \otimes_\rho Y, \tag{3.4}$$

where Y is a r.v. on N_0, independent of $U \sim U(0,1)$ and $\rho \in [0,1]$ is a constant. Obviously, a $(0, \alpha)$-GM r.v. X is α-monotone, i.e., it satisfies (1.2).

Theorem 3.7. Let X be a (ρ, α)-GM r.v. and $v \in [0,1]$ be a constant. Then, $v \otimes_{\rho'} X$ is (ρ_0, α)-GM, where $\rho_0 = \dfrac{\rho'(1-\rho)+v\rho}{1-\rho(1-v)}$ is a real value in [0, 1].

Proof. Since X is a (ρ, α)-GM, we have $X =^d U^{1/\alpha} \otimes_\rho Y$, where Y is a r.v. on N_0 independent of $U \sim U(0,1)$. Hence

$$\begin{aligned}
v \otimes_{\rho'} X &=^d v \otimes_{\rho'} (U^{1/\alpha} \otimes_\rho Y) \\
&=^d \frac{v}{1-\rho(1-v)} U^{1/\alpha} \otimes_{\rho_0} Y && \text{(by Theorem 3.2)} \\
&=^d U^{1/\alpha} \otimes_{\rho_0} \left(\frac{v}{1-\rho(1-v)} \otimes Y \right) && \text{(by Theorem 3.3(b))} \\
&=^d U^{1/\alpha} \otimes_{\rho_0} Z,
\end{aligned}$$

where $Z = \dfrac{v}{1-\rho(1-v)} \otimes Y$ is a r.v. on N_0 independent of U. Thus, by definition $v \otimes_{\rho'} X$ is (ρ_0, α)-GM with $\rho_0 = \dfrac{\rho'(1-\rho)+v\rho}{1-\rho(1-v)}$ a real value in [0, 1]. ■

Corollary 3.8. Let X be α-monotone. Then, $v \otimes_\rho X$ is a (ρ, α)-GM r.v., for all $v \in [0,1]$.

Theorem 3.9. A r.v. X with distribution $\{p_n\}_0^\infty$ is (ρ, α)-GM iff $X =^d \sum_{i=1}^{N} Z_i$, where $Z_1, Z_2, \ldots$ are i.i.d. $\mathrm{Ge}_1(1-\rho)$ r.v.'s, independent of nonnegative integer-valued α-monotone r.v. N.

Proof. Since if X satisfies (3.4) then, by (3.3), we have

$$\begin{aligned} X &=^d U^{1/\alpha} \otimes_\rho Y \\ &=^d \sum_{i=1}^{U^{1/\alpha}\otimes Y} Z_i \\ &=^d \sum_{i=1}^{N} Z_i \end{aligned}$$

where $N =^d U^{1/\alpha} \otimes Y$ is an α-monotone r.v. and vice versa. ■

Using (1.3) and Theorems 3.4, we can easily show that

Corollary 3.10. A r.v. X on N_0 is (ρ, α)-GM iff its pgf has the form

$$P_X(t) = \alpha \left(\frac{1-t}{1-\rho t}\right)^{-\alpha} \int_{1-\frac{1-t}{1-\rho t}}^{1} (1-w)^{\alpha-1} Q(w) dw,$$

where $Q(.)$ is the pgf of a r.v. on N_0.

Using a similar argument as in Theorem 2.7, we can prove analogously that the following convolution property for ρ-generalized discrete α-monotone distributions holds.

Theorem 3.11. Convolution of a (ρ, α)-GM and a (ρ, β)-GM distribution is $(\rho, \alpha+\beta)$-GM.

Example 3.12. Since $\text{Ge}_0(\pi)$ is monotone and IGe(π, ρ) is a $\text{Ge}_0(\pi)$-compounding of i.i.d. $\text{Ge}_1(1-\rho)$ r.v.'s with $\rho \in [0, 1]$, it follows that IGe(π, ρ) is a (ρ, α)-GM distributed r.v., for all $\pi \in (0, 1)$ and $\alpha \geq 1$.

In the following theorem, we give an alternative characterization to that of Alzaid and Al-Osh (1990).

Theorem 3.13. A discrete r.v. X on N_0 is α-monotone, if for every nonnegative bounded measurable function g and some $\rho \in [0, 1]$, $t^\alpha E[g(t \otimes_\rho X)]$ is non-decreasing in $t \in (0, 1]$.

Proof. Observe that for any non-negative bounded measurable g we can write

$$t^\alpha E(g(t \otimes_\rho X)) = t^\alpha E(g^*(t \otimes X)),$$

where $g^*(x) = g\left(\sum_{i=1}^{x} Z_i\right)$ belongs to the same class as g and $Z_1, Z_2, \ldots$ are i.i.d. $\text{Ge}_1(1-\rho)$ r.v.'s. Thus, the assertion follows by the result of Alzaid and Al-Osh (1990). ■

In Corollary 3.10, a (ρ, α)-GM r.v. was proved to be an N-compounding of i.i.d. $\mathrm{Ge}_1(1-\rho)$ r.v.'s, with nonnegative integer-valued α-monotone r.v. N. Thus, it is reasonable to define discrete ρ-generalized α-unimodality as follows.

Definition 3.14. A r.v. X is called ρ-generalized α-unimodal ((ρ, α)-GU), if X is an N-compounding of i.i.d. $\mathrm{Ge}_1(1-\rho)$ r.v.'s with nonnegative integer-valued α-unimodal r.v. N.

Remark 3.15. By Theorem 3.9 and Definition 3.7, it is easy to see that (ρ, α)-GU(GM) distribution is (ρ, β)-GU(GM), provided that $\beta \geq \alpha$. Also, the convolution of a (ρ, α)-GU and a (ρ, β)-GU distribution is $(\rho, \alpha+\beta)$-GU.

Example 3.16. Since Poisson distribution with parameter λ, denoted by $\mathrm{Po}(\lambda)$, is a unimodal r.v. and $\mathrm{IPo}(\lambda, \rho)$ is a $\mathrm{Po}(\lambda)$-compounding of i.i.d. $\mathrm{Ge}(1-\rho)$ r.v.'s with $\rho \in [0, 1]$, it follows that $\mathrm{IPo}(\lambda, \rho)$ is a (ρ, α)-GU distributed r.v., for all $\lambda \in (0, \infty)$ and $\alpha \geq 1$. Similarly, $\mathrm{INB}(r, \pi, \rho)$ and $\mathrm{IBin}(n, \pi, \rho)$ are (ρ, α)-GU distributed r.v., for all $\pi \in (0, 1)$ and $\alpha \geq 1$.

4. ρ-Inflated Discrete Self-Decomposable Distributions

It is known that a discrete r.v. X on N_0 is self-decomposable if, for any $\alpha \in (0, 1)$, there exists a r.v. X_α independent of X such that

$$X =^d \alpha \otimes X + X_\alpha \tag{4.1}$$

or equivalently, if

$$P_X(t) = P_X(1-\alpha(1-t))P_\alpha(t),$$

for some pgf P_α.

It has been shown that discrete self-decomposable r.v.'s are α-unimodal ($\alpha \geq 1$) and infinitely divisible. For more details, we refer the readers to Steutel and van Harn (2004). Here we shall extend this notion using our operation $\otimes_\rho$ as follows.

Definition 4.1. A r.v. X is called ρ-inflated discrete self-decomposable (ρ-ISD), if X is an N-compounding of i.i.d. $\mathrm{Ge}_1(1-\rho)$ r.v.'s with nonnegative integer-valued self-decomposable r.v. N.

If we denote such r.v. by X_N, we see that, in fact X_N is a ρ-ISD r.v. if, for any $\alpha \in (0, 1)$, there exists a r.v. N_α independent of N such that

$$X_N =^d \alpha \otimes_\rho N + 1 \otimes_\rho N_\alpha.$$

In terms of pgf's, a r.v. X_N with pgf P is called ρ-ISD, if for any $\alpha \in (0, 1)$, there exists a pgf P_α such that

$$\begin{aligned}
P_{X_N}(t) &= P_N(P_G(t)) \\
&= P_{\alpha\otimes N+N_\alpha}\left(\frac{t(1-\rho)}{1-\rho t}\right) && \text{(by (4.1))} \\
&= P_{\alpha\otimes N}\left(\frac{t(1-\rho)}{1-\rho t}\right).P_\alpha(t), && (4.2)
\end{aligned}$$

where $P_G(t) = \dfrac{(1-\rho)t}{1-\rho t}$, with the inverse function $P_G^{-1}(t) = \dfrac{t}{1-\rho+\rho t}$, and $P_N(.)$ is the pgf of r.v. N. It follows that

$$P_X \circ P_G^{-1}(t) = P_N(t). \tag{4.3}$$

Therefore, N in Definition 4.1, is a r.v. with pgf $P_X \circ P_G^{-1}$. This implies that the following theorem is valid.

Theorem 4.2. The pgf P_X is ρ-ISD iff $P_X \circ P_G^{-1}$ is a self-decomposable pgf, where $P_G^{-1}(t) = \dfrac{t}{1-\rho+\rho t}$.

Example 4.3. All self-decomposable-compounding of i.i.d. $\mathrm{Ge}_1(1-\rho)$ r.v.'s with $\rho \in (0, 1)$ are ρ-ISD. For example, IPo(λ, ρ) and INB(r, π, ρ) are ρ-ISD r.v.'s, for all $\lambda \in (0, \infty)$ and $\pi \in (0, 1)$, but IBin(n, π, ρ) (with pmf (1.7)) is not, because binomial distributions are not self-decomposable.

By Definition 4.1, it is easy to show that ρ-inflated self-decomposability is closed under convolution and the following theorems are also valid.

Theorem 4.4. A ρ-ISD r.v. is (ρ, α)-GU, $\alpha \geq 1$.

Theorem 4.5. A ρ-ISD r.v. X is infinitely divisible.

Finally, the following theorem gives an analogue characterization to (4.1).

Theorem 4.6. Let X be a ρ-ISD r.v., then for all $\alpha \in (0, 1)$, there exists a r.v. X_α such that

$$X =^d \beta \otimes_{\rho_0} X + X_\alpha, \tag{4.4}$$

where $\beta = \dfrac{\alpha(1-\rho)}{1-\alpha\rho}$ and $\rho_0 = \dfrac{\rho(1-\alpha)}{1-\rho\alpha}$ are real values in [0, 1].

Proof. By (4.3), we have $P_N(t) = P_X\left(\dfrac{t}{1-\rho+\rho t}\right)$. Then,

$$\begin{aligned}
P_{\alpha\otimes N}(t) &= P_N(1-\alpha+\alpha t) \\
&= P_X\left(\frac{1-\alpha+\alpha t}{1-\alpha\rho+\alpha\rho t}\right).
\end{aligned}$$

Hence, from (4.1), we have

$$\begin{aligned}
P_X(t) &= P_{\alpha\otimes N}(P_G(t)).P_\alpha(t) \\
&= P_X\left(\frac{1-\alpha+\alpha P_G(t)}{1-\alpha\rho+\alpha\rho P_G(t)}\right).P_\alpha(t) \\
&= P_X\left(\frac{1-\alpha-(\rho-\alpha)t}{1-\alpha\rho-\rho(1-\alpha)t}\right).P_\alpha(t) \\
&= P_X\left(1-\beta+\beta\frac{(1-\rho_0)t}{1-\rho_0 t}\right).P_\alpha(t) \\
&= P_{\beta\otimes X}\left(\frac{(1-\rho_0)t}{1-\rho_0 t}\right).P_\alpha(t) \\
&= P_{\beta\otimes_{\rho_0}X}(t).P_\alpha(t)\,.
\end{aligned}$$

This implies (4.4), as required. ■

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