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On the Integral homology and counterexamples to the Andreotti-Grauert conjecture

Mohammed Mouçouf and Youssef Alaoui

Département de mathématiques et Informatique,
Université Chouaib, Doukkali,
Faculté des Sciences, B.P.20; 24000,
El Jadida, Morocco.
E-mail: moucouf@hotmail.com,
y.alaoui@iav.ac.ma or comp5123ster@gmail.com

Abstract

In this paper, we prove by means of a counterexample that there exist pair of integers (n, p) with $n \geq 3$, $1 < p \leq n-1$, and open sets D in $\mathbb{C}^n$ which are cohomologically p-complete with respect to the structure sheaf $\mathcal{O}_D$ such that the cohomology group $H_{n+p}(D, \mathbb{Z})$ does not vanish. In particular D is not p-complete.

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1. Introduction

By the theory of Andreotti and Grauert [1] it is known that a q-complete complex space is always cohomologically q-complete.

It was shown in [6] that if X is a Stein manifold and Ω is a cohomologically q-complete open set in X with respect to the structure sheaf $\mathcal{O}_\Omega$, then Ω is q-complete, if it has a smooth boundary.

If now X is a q-complete space of complex dimension n, then $H_p(X, \mathbb{Z}) = 0$ for $p \geq n+q$ and $H_{n+q-1}(X, \mathbb{Z})$ is free (see [5] and [11]). Considering the short exact sequence given by the universal coefficient theorem

$$0 \to Ext_{\mathbb{Z}}(H_{p-1}(X, \mathbb{Z}), \mathbb{Z}) \to H^p(X, \mathbb{Z}) \to Hom_{\mathbb{Z}}(H_p(X, \mathbb{Z}), \mathbb{Z}) \to 0$$

we see that $H^p(X, \mathbb{Z}) = 0$ for $p \geq n+q$.

If X is a cohomologically q-complete complex space of dimension n, it follows from [9] that $H_p(X, \mathbb{C}) = 0$ for all integers $p \geq n + q$. Since $H^p(X, \mathbb{C})$, for an n-dimensional complex manifold X, are Frechet spaces whose topological duals are isomorphic to $H_p(X, \mathbb{C})$ (see [10]), then X satisfies the topological property $H^p(X, \mathbb{C}) = 0$ for $p \geq n + q$.

However, it seems unknown if $H^p(X, \mathbb{Z})$, $p \geq n + q$, vanishes, if X is assumed to be only cohomologically q-complete.

In this article, we prove that there exist pair of integers (n, p) with $1 < p \leq n - 1$, and cohomologically p-complete open sets D with respect to $\mathcal{O}_D$ in $\mathbb{C}^n$ such that $H_{n+p}(D, \mathbb{Z}) \neq 0$. It is clear that D is not p-complete.

2. Preliminaries

Let Ω be a an open set in $\mathbb{C}^n$ with complex coordinates $z_1, \cdots, z_n$. Then it is known that a function $\phi \in C^\infty(\Omega)$ is said to be q-convex if the hermetian form $L_z(\phi, \xi) = \sum_{i,j} \frac{\partial^2 \phi(z)}{\partial z_i \partial \overline{z}_j} \xi_i \overline{\xi}_j$ has at least $n - q + 1$ positive eingenvalues at each point $z \in \Omega$.

A function $\rho \in C^o(\Omega, \mathbb{R})$ is called q-convex with corners if ρ is locally a maximum of finite number of q-convex functions.

We say that Ω is q-complete (resp. q-complete with corners) if there exists an exhaustion function ϕ on Ω which is q-convex (resp. q-convex with corners).

An open subset D of Ω is called q-Runge if for every compact set $K \subset D$, there is a q-convex exhaustion function $\phi \in C^\infty(\Omega)$ such that

$$K \subset \{x \in \Omega : \phi(x) < 0\} \subset\subset D$$

It is shown in [1] that if Ω is q-complete, then Ω is cohomologically q-complete, which means that for every coherent analytic sheaf $\mathcal{F}$ on Ω and $p \geq q$, the cohomology groups $H^p(\Omega, \mathcal{F})$ vanish. Moreover, if D is q-Runge in Ω, then for every $\mathcal{F} \in coh(\Omega)$ the restriction map

$$H^p(\Omega, \mathcal{F}) \to H^p(D, \mathcal{F})$$

has a dense image for all $p \geq q - 1$.

The purpose of the present article is to prove the following theorem

Theorem 2.1. Let (n, q) be a pair of integers such that $n \geq 3$, $1 < q < n$, and q does not divide n. We put $m = \left[\frac{n}{q}\right]$ and $r = n - mq$. Then there exists a cohomologically $(\tilde{q}-3)$-complete domain D in $\mathbb{C}^n$ such that $H_{n+\tilde{q}-3}(D, \mathbb{Z}) \neq 0$, if $r = 1$ and $m > q \geq 2$. Here $\tilde{q} = n - \left[\frac{n}{q}\right] + 1$ and $\left[\frac{n}{q}\right]$ is the integral part of $\frac{n}{q}$.

3. Proof of the theorem

Let (n, q) be a pair of integers with $n \geq 3$ and $1 \leq q \leq n$. We put $m = \left[\frac{n}{q}\right]$ and suppose that $r = n - mq > 0$. We consider the functions $\phi_1, \cdots, \phi_{m+1}$ defined on $\mathbb{C}^n$ by

$$\phi_j = \sigma_j + \sum_{i=1}^{m} \sigma_i^2 - \frac{1}{4}||z||^2 + N||z||^4, \quad j = 1, \cdots, m,$$

and

$$\phi_{m+1} = -\sigma_1 - \cdots - \sigma_m + \sum_{i=1}^{m} \sigma_i^2 - \frac{1}{4}||z||^2 + N||z||^4,$$

where $\sigma_j = y_j + \sum_{i=m+1}^{n} |z_i|^2 - (m+1)(\sum_{i=m+(j-1)(q-1)+1}^{m+j(q-1)} |z_i|^2)$ for $j = 1, \cdots, m$, $z_j = x_j + iy_j$. Then it is known from [4] that all ϕ_j, $1 \leq j \leq m+1$, are q-convex on $\mathbb{C}^n$, if $N > 0$ is sufficiently large and, if $\rho = Max\{\phi_j, 1 \leq j \leq m+1\}$, then there exists a small constant $\varepsilon_o > 0$ such that the set $D_{\varepsilon_o} = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon_o\}$ is relatively compact in the unit ball $B = B(0, 1)$, if N is large enough.

Let now $\varepsilon > \varepsilon_0$ and choose Stein open sets $U_1, \cdots, U_k \subset\subset D_{\varepsilon_o}$ covering ∂D_ε and functions $\theta_j \in C_o^\infty(U_j, \mathbb{R}^+)$ such that $\sum_{j=1}^{k} \theta_j(x) > 0$ at any point $x \in \partial D_\varepsilon$. There exist sufficiently small constants $c_1 > 0, \cdots, c_k > 0$ such that the functions $\phi_{i,j} = \phi_i - \sum_{i=1}^{j} c_i\theta_i$ are q-convex for $i = 1, \cdots, m+1$ and $1 \leq j \leq k$. We define $\phi_{i,0} = \phi_i$ for $i = 1, \cdots, m+1$, $D_o = D_\varepsilon$ and $D_j = \{z \in D_{\varepsilon_o} : \rho_j(z) < -\varepsilon\}$, where

$$\rho_j = \rho - \sum_{i=1}^{j} c_i\theta_i, \quad j = 1, \cdots, k$$

Then ρ_j are q-convex with corners, $D_o \subset D_1 \subset \cdots \subset D_k$, $D_o \subset\subset D_k \subset\subset D_{\varepsilon_o}$ and $D_j \setminus D_{j-1} \subset\subset U_j$ for $j = 1, \cdots, k$.

Lemma 3.1. Let $\mathcal{F}$ be a coherent analytic sheaf on D_{ε_0}. Then the restriction map $H^p(D_{j+1}, \mathcal{F}) \to H^p(D_j, \mathcal{F})$ is surjective for every $p \geq \tilde{q} - 2$ and all $0 \leq j \leq k-1$. In particular, $dim_{\mathbb{C}} H^p(D_j, \mathcal{F}) < \infty$ for $p \geq \tilde{q} - 2$ and $j = 0, \cdots, k$.

Proof. We first prove that for every $p \geq \tilde{q} - 2$, $H^p(D_j \cap U_l, \mathcal{F}) = 0$ for all $0 \leq j \leq k$ and $1 \leq l \leq k$. We fix $j \in \{0, \cdots, k\}$ and we write $D_j \cap U_l = D_1' \cap \cdots \cap D_{m+1}'$, where $D_i' = \{z \in U_l : \phi_{i,j}(z) < -\varepsilon\}$ are clearly q-complete and q-Runge in U_l. Then for any integer $t \leq m$, $D_{i_1}' \cap \cdots \cap D_{i_t}'$ is $(\tilde{q}-1)$-Runge in U_l for all $i_1, \cdots, i_t \in \{1, \cdots, m+1\}$.

In fact, let $K \subset D'_{i_1} \cap \cdots \cap D'_{i_t}$ be an arbitrary compact subset. There exists for any $i \in \{i_1, \cdots, i_t\}$ a q-convex exhaustion function ψ_i on U_l such that

$$K \subset \{x \in U_l : \psi_i(x) < 0\} \subset\subset D'_i$$

Define $\psi = Max(\psi_{i_1}, \cdots, \psi_{i_t})$. Then $K \subset \{x \in U_l : \psi(x) < 0\} \subset\subset D'_{i_1} \cap \cdots \cap D'_{i_t}$ and, ψ can be approximated in the C^O-topology by smooth $(t(q-1)+1)$-convex functions (see [10]). Since q does not divide n and $t \leq m$, then $t(q-1)+1 \leq \tilde{q}-1$. Therefore a suitable smooth $(\tilde{q}-1)$-convex approximation of ψ shows that $D'_{i_1} \cap \cdots \cap D'_{i_t}$ is $(\tilde{q}-1)$-Runge in U_l. This implies that for $r \geq \tilde{q}-2$, the image of $H^r(U_l, \mathcal{F})$ is dense in the cohomology group $H^r(D'_{i_1} \cap \cdots \cap D'_{i_t}, \mathcal{F})$ which is separated, since $\tilde{q}-2 \geq 2$ ($n \geq 4$ and $q \nmid n$). Thus $H^r(D'_{i_1} \cap \cdots \cap D'_{i_t}, \mathcal{F}) = 0$ for all $r \geq \tilde{q}-2$. Then, by ([8], Proposition 1), we have

$$H^r(D_j \cap U_l, \mathcal{F}) \cong H^{r+m}(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{F}) \;\; if \;\; r \geq \tilde{q}-2$$

Since $U_l \backslash D'_1 \cup \cdots \cup D'_{m+1}$ has no compact connected components, then $D'_1 \cup \cdots \cup D'_{m+1}$ is n-Runge in U_l (see [3]). But for $r \geq \tilde{q}-2$, $r+m \geq n-1$ and hence $H^{r+m}(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{F}) = 0$.

Now since $D_{j+1} = D_j \cup (D_{j+1} \cap U_{j+1})$, then the Mayer-Vietoris sequence for cohomology

$$\begin{aligned} \cdots \to H^p(D_{j+1}, \mathcal{F}) \to H^p(D_j, \mathcal{F}) \oplus H^p(D_{j+1} \cap U_{j+1}, \mathcal{F}) \\ \to H^p(D_j \cap U_{j+1}, \mathcal{F}) \to \cdots \end{aligned}$$

implies that the restriction map $H^p(D_{j+1}, \mathcal{F}) \to H^p(D_j, \mathcal{F})$ is surjective for all $p \geq \tilde{q}-2$. ∎

We now choose n and q such that $m = \left[\frac{n}{q}\right] \geq q$ and $r = n - mq \geq 1$, and let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on B. Then we have the following

Lemma 3.2. The restriction map $H^p(D_{\varepsilon_o}, \mathcal{O}) \to H^p(D_\varepsilon, \mathcal{O})$ has dense image for every $p \geq \tilde{q}-3$ and $\varepsilon \geq \varepsilon_o$.

Proof. Let T be the set of all real numbers $\varepsilon \geq \varepsilon_o$ such that $H^p(D_\varepsilon, \mathcal{O}) \to H^p(D_{\varepsilon_1}, \mathcal{O})$ has a dense image for every real number $\varepsilon_1 > \varepsilon$ and all $p \geq \tilde{q}-3$. Then $T \neq \emptyset$. In fact, choose $\varepsilon > \varepsilon_o$ such that $-\varepsilon < Min_{\overline{B} \backslash D_{\varepsilon_o}}\{\phi_i(z), i = 1, \cdots, m+1\}$, and let $\varepsilon_1 > \varepsilon$. If $D_{\varepsilon_1} \neq \emptyset$, then $D_i = \{z \in B : \phi_i(z) < -\varepsilon_1\}$ and $D'_i = \{z \in B : \phi_i(z) < -\varepsilon\}$ are relatively compact in D_{ε_o}, q-complete and q-Runge in B.

Note first that for every $i_1, \cdots, i_{m-1} \in \{1, \cdots, m+1\}$, the cohomology group $H^p(D_{i_1} \cap \cdots \cap D_{i_{m-1}}, \mathcal{O}) = 0$ for $p \geq \tilde{q}-2$, since $D_{i_1} \cap \cdots \cap D_{i_{m-1}}$ is $((m-1)(q-1)+1)$-complete and $\tilde{q}-2 \geq ((m-1)(q-1)+1)$. Next, we show, exactly as in the proof of lemma 1, that for every $i_1, \cdots, i_m \in \{1, \cdots, m+1\}$, $D_{i_1} \cap \cdots \cap D_{i_m}$ is $(\tilde{q}-1)$-Runge in B, which implies that the restriction map $H^p(D_{i_1} \cap \cdots \cap D_{i_{m-1}}, \mathcal{O}) \to H^p(D_{i_1} \cap \cdots \cap$

$D_{i_m}, \mathcal{O})$ has a dense image for all $p \geq \tilde{q}-2$. This shows that $H^p(D_{i_1} \cap \cdots \cap D_{i_m}, \mathcal{O}) = 0$ for $p \geq \tilde{q} - 2$. Then, by using ([8], Proposition 1), we obtain

$$H^p(D_{\varepsilon_1}, \mathcal{O}) \cong H^{p+m}(D_1 \cup \cdots \cup D_{m+1}, \mathcal{O}) \text{ for } p \geq \tilde{q} - 2.$$

Similarly

$$H^p(D_{\varepsilon}, \mathcal{O}) \cong H^{p+m}(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{O}) \text{ for } p \geq \tilde{q} - 2.$$

Since $D_1 \cup \cdots \cup D_{m+1}$ is n-Runge in B and contained in the open set $D'_1 \cup \cdots \cup D'_{m+1} \subset\subset D_{\varepsilon_0}$, then the restriction map

$$H^p(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{O}) \rightarrow H^p(D_1 \cup \cdots \cup D_{m+1}, \mathcal{O})$$

has a dense image for $p \geq n-1$, which means that $H^p(D_\varepsilon, \mathcal{O}) \rightarrow H^p(D_{\varepsilon_1}, \mathcal{O})$ has a dense image for $p \geq \tilde{q} - 2$. We are now going to show that $H^{\tilde{q}-3}(D_\varepsilon, \mathcal{O}) = H^{\tilde{q}-3}(D_{\varepsilon_1}, \mathcal{O}) = 0$. To see this, let $\Omega_m = \{z \in D_1 \cap \cdots \cap D_m : \phi_{m+1}(z) > -\varepsilon\}$ and $S_m = \{z \in D_1 \cap \cdots \cap D_m : \phi_{m+1}(z) \leq -\varepsilon\}$. Then $H^p_{S_m}(D_1 \cap \cdots \cap D_m, \mathcal{O}) = 0$ for all $p \leq n-q$, where $H^j_{S_m}(D_1 \cap \cdots \cap D_m, \mathcal{O})$ is the j-th group of cohomology of $D_1 \cap \cdots \cap D_m$ with support in S_m. In fact, for each point $\xi \in D_1 \cap \cdots \cap D_m$ there exists [1] a fundamental system of connected Stein neighborhoods $U \subset D_1 \cap \cdots \cap D_m$ of ξ such that $H^j(U \cap \Omega_m, \mathcal{O}) = 0$ for $0 < j < n-q$ and, the restriction map

$$\Gamma(U, \mathcal{O}) \longrightarrow \Gamma(U \cap \Omega_m, \mathcal{O})$$

is an isomorphism. It follows from [7] that $\underline{H^j_{S_m}}(\mathcal{O}) = 0$ for $0 \leq j \leq n-q$, where $\underline{H^j_{S_m}}(\mathcal{O})$ is the cohomology sheaf of $D_1 \cap \cdots \cap D_m$ with coefficient in $\mathcal{O}$ and support in S_m. By ([7]) there is a spectral sequence

$$H^p_{S_m}(D_1 \cap \cdots \cap D_m, \mathcal{O}) \Longleftarrow E_2^{p,q} = H^p(D_1 \cap \cdots \cap D_m, \underline{H^q_{S_m}}(\mathcal{O}))$$

Since $\underline{H^p_{S_m}}(\mathcal{O}) = 0$ for $p \leq n-q$, then the abutment $H^p_{S_m}(D_1 \cap \cdots \cap D_m, \mathcal{O}) = 0$ for $p \leq n-q$.

Now it follows from the exact sequence of local cohomology

$$\begin{aligned} \cdots \rightarrow H^p_{S_m}(D_1 \cap \cdots \cap D_m, \mathcal{O}) &\rightarrow H^p(D_1 \cap \cdots \cap D_m, \mathcal{O}) \\ \rightarrow H^p(\Omega_m, \mathcal{O}) &\rightarrow H^{p+1}_{S_m}(D_1 \cap \cdots \cap D_m, \mathcal{O}) \rightarrow \cdots \end{aligned}$$

that $H^p(D_1 \cap \cdots \cap D_m, \mathcal{O}) \cong H^p(\Omega_m, \mathcal{O})$ for $0 \leq p \leq n-q-1$.

Note that since $\Omega_m = \{z \in (D_1 \cap \cdots \cap D_m) \cup D_{m+1} : \phi_{m+1}(z) > -\varepsilon\}$, one can verify exactly as for $H^p_{S_m}(D_1 \cap \cdots \cap D_m, \mathcal{O})$ that, if $S'_m = \{z \in (D_1 \cap \cdots \cap D_m) \cup D_{m+1} : \phi_{m+1}(z) \leq -\varepsilon\} = D_{m+1} \cup S_m$, then $H^p_{S'_m}((D_1 \cap \cdots \cap D_m) \cup D_{m+1}, \mathcal{O}) = 0$ for $p \leq n-q$ and, therefore

$$\begin{aligned} H^p((D_1 \cap \cdots \cap D_m) \cup D_{m+1}, \mathcal{O}) &\cong H^p((D_1 \cap \cdots \cap D_m) \cup D_{m+1} \setminus S'_m, \mathcal{O}) \\ &= H^p(\Omega_m, \mathcal{O}) \cong H^p(\Omega_m \cup D_{m+1}, \mathcal{O}) \end{aligned}$$

for $q \leq p \leq n-q-1$.

Now, since, in addition, $\Omega_m \cap D_{m+1} = \emptyset$, then, by using the Mayer-Vietoris sequence for cohomology

$$\begin{aligned}
\cdots &\to H^p((D_1 \cap \cdots \cap D_m) \cup D_{m+1}, \mathcal{O}) \\
&\to H^p((D_1 \cap \cdots \cap D_m), \mathcal{O}) \oplus H^p(D_{m+1}, \mathcal{O}) \\
&\to H^p(D_\varepsilon, \mathcal{O}) \to H^{p+1}((D_1 \cap \cdots \cap D_m) \cup D_{m+1}, \mathcal{O}) \\
&\to H^{p+1}((D_1 \cap \cdots \cap D_m), \mathcal{O}) \oplus H^{p+1}(D_{m+1}, \mathcal{O}) \to \cdots
\end{aligned}$$

and the fact that D_{m+1} is q-complete, we find that $H^p(D_\varepsilon, \mathcal{O}) = 0$ for $q \leq p \leq n-q-2$. This implies that $H^{\tilde{q}-3}(D_\varepsilon, \mathcal{O}) = 0$ and similarly $H^{\tilde{q}-3}(D_{\varepsilon_1}, \mathcal{O}) = 0$, which proves that $\varepsilon \in T$.

To see that T is open in $[\varepsilon_o, +\infty[$, it is sufficient to prove that if $\varepsilon \in T$, $\varepsilon > \varepsilon_o$, then there is $\varepsilon_o < \varepsilon' < \varepsilon$ such that $\varepsilon' \in T$. For this, we consider as in lemma 1, finitely many Stein open sets $U_i \subset\subset D_{\varepsilon_o}$, $i = 1, \cdots, k$, such that $\partial D_\varepsilon \subset \bigcup_{i=1}^{k} U_i$ and functions $\theta_j \in C^\infty(U_j, \mathbb{R}^+)$ with compact support such that $\sum_{j=1}^{k} \theta_j(x) > 0$ at any point $x \in \partial D_\varepsilon$. Next we define $D_j(\varepsilon) = \{z \in D_{\varepsilon_o} : \rho_j(z) < -\varepsilon\}$, where $\rho_j(z) = Max(\phi_1 - \sum_{i=1}^{j} c_i\theta_i, \cdots, \phi_{m+1} - \sum_{i=1}^{j} c_i\theta_i)$ with $c_i > 0$ sufficiently small so that the functions $\phi_i - \sum_{i=1}^{j} c_i\theta_i$ are q-convex for $1 \leq i \leq m+1$ and $1 \leq j \leq k$. Then, by lemma 1, the restriction map $H^p(D_k(\varepsilon), \mathcal{O}) \to H^p(D_\varepsilon, \mathcal{O})$ is surjective for $p \geq \tilde{q}-2$ and, there exists $\varepsilon_o < \varepsilon' < \varepsilon$ such that $D_\varepsilon \subset D_{\varepsilon'} \subset D_k(\varepsilon)$. If now $\varepsilon' \leq \alpha \leq \varepsilon$, then we have

$$D_\alpha \subset D_{\varepsilon'} \subset D_k(\varepsilon) \subset D_k(\alpha) \subset\subset D_{\varepsilon_o}.$$

Since, by lemma 1, the restriction map $H^p(D_k(\alpha), \mathcal{O}) \to H^p(D_\alpha, \mathcal{O})$ is surjective for $p \geq \tilde{q}-2$ and $H^{\tilde{q}-3}(D_\alpha, \mathcal{O}) = 0$, then $H^p(D_{\varepsilon'}, \mathcal{O}) \to H^p(D_\alpha, \mathcal{O})$ is surjective for $p \geq \tilde{q}-3$. For $\alpha > \varepsilon$, we have $D_\alpha \subset D_\varepsilon \subset D_{\varepsilon'} \subset D_k(\varepsilon)$. Since $H^p(D_k(\varepsilon), \mathcal{O}) \to H^p(D_\varepsilon, \mathcal{O})$ is surjective for $p \geq \tilde{q}-3$, and $H^p(D_\varepsilon, \mathcal{O}) \to H^p(D_\alpha, \mathcal{O})$ has a dense image for $p \geq \tilde{q}-3$, then $H^p(D_{\varepsilon'}, \mathcal{O}) \to H^p(D_\alpha, \mathcal{O})$ has a dense image for $p \geq \tilde{q}-3$, which implies that $\varepsilon' \in T$.

In order to prove that T is closed, we consider a sequence of real numbers $\varepsilon_j \in T$, $j \geq 0$, such that $\varepsilon_j \searrow \varepsilon$ and a Stein open covering $\mathcal{U} = (U_i)_{i \in I}$ of D_{ε_o} with a countable base of open subsets of D_{ε_o}. We fix $p \geq \tilde{q}-3$. Then the restriction map of spaces of cocycles $Z^p(\mathcal{U}|_{D_{\varepsilon_{j+1}}}, \mathcal{O}) \to Z^p(\mathcal{U}|_{D_{\varepsilon_j}}, \mathcal{O})$ has a dense image for $j \geq 0$. Therefore, by ([1], p. 246), the restriction map $Z^p(\mathcal{U}|_{D_\varepsilon}, \mathcal{O}) \to Z^p(\mathcal{U}|_{D_{\varepsilon_j}}, \mathcal{O})$ has also a dense image. Let $\varepsilon' > \varepsilon$ and $j \in \mathbb{N}$ such that $\varepsilon' > \varepsilon_j$. Since $\varepsilon_j \in T$, then

$Z^p(\mathcal{U}|_{D_{\varepsilon_j}}, \mathcal{O}) \to Z^p(\mathcal{U}|_{D_{\varepsilon'}}, \mathcal{O})$ has a dense image, and hence $\varepsilon \in T$. This completes the proof of lemma 3.3. ■

Let now A be the set of all real numbers $\varepsilon \geq \varepsilon_0$ such that $H^p(D_\varepsilon, \mathcal{O}) = 0$ for all $p \geq \tilde{q} - 2$. Then in the situation described above we have

Lemma 3.3. The set A is not empty and, for every $\varepsilon \in A$ with $\varepsilon > \varepsilon_0$, there is $\varepsilon_o \leq \varepsilon' < \varepsilon$ such that $\varepsilon' \in A$.

Proof. Let $\varepsilon_1 > \varepsilon_o$ be such that

$$-\varepsilon_1 < Inf_{z \in \partial D_{\varepsilon_o}}\{\phi_i(z), i = 1, 2, \cdots, m+1\}.$$

Then $[\varepsilon_1, +\infty[\subset A$. In fact, let $\varepsilon \geq \varepsilon_1$ and write $D_\varepsilon = D_{\varepsilon,1} \cap D_{\varepsilon,2} \cap \cdots \cap D_{\varepsilon,m+1}$, where $D_{\varepsilon,i} = \{z \in B : \phi_i(z) < -\varepsilon\} \subset\subset D_{\varepsilon_0}$ is q-complete and q-Runge in B for all $i \in \{1, \cdots, m+1\}$. Moreover, for every integers $t \leq m$ and $i_1, \cdots, i_t \in \{1, \cdots, m+1\}$, $D_{\varepsilon,i_1} \cap D_{\varepsilon,i_2} \cap \cdots \cap D_{\varepsilon,i_t}$ is $(tq - t + 1)$-Runge in B and $\tilde{q} - 2 \geq tq - t$, then $D_{\varepsilon,i_1} \cap D_{\varepsilon,i_2} \cap \cdots \cap D_{\varepsilon,i_t}$ is cohomologically $(\tilde{q} - 2)$-complete. Therefore, by ([8], Proposition 1)

$$H^r(D_\varepsilon, \mathcal{O}) \cong H^{r+m}(D_{\varepsilon,1} \cup D_{\varepsilon,2} \cup \cdots \cup D_{\varepsilon,m+1}, \mathcal{O}), \quad for \ \ r \geq \tilde{q} - 2$$

Since $D_{\varepsilon,1} \cup D_{\varepsilon,2} \cup \cdots \cup D_{\varepsilon,m+1}$ is n-Runge in B, then $H^p(B, \mathcal{O}) \to H^p(D_{\varepsilon,1} \cup D_{\varepsilon,2} \cup \cdots \cup D_{\varepsilon,m+1}, \mathcal{O})$ has a dense image if $p \geq n-1$. Hence $H^p(D_{\varepsilon,1} \cup D_{\varepsilon,2} \cup \cdots \cup D_{\varepsilon,m+1}, \mathcal{O})$ vanishes for $p \geq n - 1$ and so is $H^p(D_\varepsilon, \mathcal{O})$ if $p \geq \tilde{q} - 2$, since $\tilde{q} - 2 + m = n - 1$.

For the proof of the second assertion of the lemma, we can write for all $0 \leq j \leq k$ and $1 \leq l \leq k$, $D_j \cap U_l = D'_1 \cap \cdots \cap D'_{m+1}$, where $D'_i = \{z \in U_l : \phi_{i,j}(z) < -\varepsilon\}$ are q-complete and q-Runge in U_l. Therefore for all $t \leq m$ and $i_1, \cdots, i_t \in \{1, \cdots, m+1\}$, $D'_{i_1} \cap D'_{i_2} \cap \cdots \cap D'_{i_t}$ is $(\tilde{q} - 1)$-Runge in U_l, which implies that $H^p(U_l, \mathcal{O})$ has a dense image in $H^p(D'_{i_1} \cap D'_{i_2} \cap \cdots \cap D'_{i_t}, \mathcal{O})$ for all $p \geq \tilde{q} - 2$. This shows that $H^p(D'_{i_1} \cap D'_{i_2} \cap \cdots \cap D'_{i_t}, \mathcal{O}) = 0$ for all $p \geq \tilde{q} - 2$. Moreover, by ([8], Proposition 1), one obtains

$$H^p(D_j \cap U_l, \mathcal{O}) \cong H^{p+m}(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{O}) \ \ for \ \ p \geq \tilde{q} - 2.$$

Since $\tilde{q} - 2 + m = n - 1$ and $D'_1 \cup \cdots \cup D'_{m+1}$ is n-Runge in U_l, then

$$H^p(D_j \cap U_l, \mathcal{O}) \cong H^{p+m}(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{O}) = 0 \ \ for \ \ p \geq \tilde{q} - 2.$$

We now consider the Mayer-Vietoris sequence for cohomology

$$\begin{aligned} \cdots \to H^{p-1}(D_j \cap U_{j+1}, \mathcal{O}) &\to H^p(D_{j+1}, \mathcal{O}) \\ &\to H^p(D_j, \mathcal{O}) \oplus H^p(D_{j+1} \cap U_{j+1}, \mathcal{O}) \\ &\to H^p(D_j \cap U_{j+1}, \mathcal{O}) \to \cdots \end{aligned}$$

Since $H^p(D_j \cap U_{j+1}, \mathcal{O}) = H^p(D_{j+1} \cap U_{j+1}, \mathcal{O}) = 0$ for $p \geq \tilde{q}-2$, then $H^p(D_{j+1}, \mathcal{O}) \to H^p(D_j, \mathcal{O})$ is surjective for all $p \geq \tilde{q} - 2$.

We are now going to show that $H^p(D_{j+1}, \mathcal{O}) \to H^p(D_j, \mathcal{O})$ is injective for all $p \geq \tilde{q} - 2$. Let $\mathcal{V} = (V_i)_{i \in \mathbb{N}}$ be an open covering of D_{ε_o} with a fundamental system of Stein neighborhoods of D_{ε_o} such that if $V_{i_o} \cap \cdots \cap V_{i_r} \neq \emptyset$ and $V_{i_o} \cup \cdots \cup V_{i_r} \subset D_{j+1}$, then $V_{i_o} \cup \cdots \cup V_{i_r} \subset D_j$ or $V_{i_o} \cup \cdots \cup V_{i_r} \subset U_{j+1} \cap D_{j+1}$.

We first show that $H^{n-1}(D_k, \mathcal{O}) = 0$. We shall prove it assuming that it has already been proved for $j < k$. For this, we consider the Mayer-Vietoris sequence for cohomology

$$\to H^{\tilde{q}-3}(D_j, \mathcal{O}) \oplus H^{\tilde{q}-3}(D_{j+1} \cap U_{j+1}, \mathcal{O}) \overset{r^*}{\to} H^{\tilde{q}-3}(D_j \cap U_{j+1}, \mathcal{O})$$
$$\overset{j^*}{\to} H^{\tilde{q}-2}(D_{j+1}, \mathcal{O}) \overset{\rho^*}{\to}$$

Let ξ be a cocycle in $Z^{\tilde{q}-2}(\mathcal{V}|_{D_{j+1}}, \mathcal{O})$ and let $\rho(\xi)$ be its restriction to a cocycle in $Z^{\tilde{q}-2}(\mathcal{V}|_{D_j}, \mathcal{O})$. Since $\rho(\xi)$ is a coboundary by induction and $H^{\tilde{q}-2}(D_{j+1} \cap U_{j+1}, \mathcal{O}) = 0$, from the Mayer-Vietoris sequence, it follows that there exist

$$\eta \in Z^{\tilde{q}-3}(\mathcal{V}|_{D_j \cap U_{j+1}}, \mathcal{O}) \text{ and } \mu \in C^{\tilde{q}-3}(\mathcal{V}|_{D_{j+1}}, \mathcal{O})$$

such that $\xi = j(\eta) + \delta\mu$. There exists a sequence $\{\eta_n\} \subset Z^{\tilde{q}-3}(\mathcal{V}|_{D_{j+1} \cap U_{j+1}}, \mathcal{O})$ with $r(\eta_n) - \eta \to 0$, when $n \to \infty$. This is possible because $Z^{\tilde{q}-3}(\mathcal{V}|_{D_{j+1} \cap U_{j+1}}, \mathcal{O}) \to Z^{\tilde{q}-3}(\mathcal{V}|_{D_j \cap U_{j+1}}, \mathcal{O})$ has a dense range. (See the proof of lemma 2). Now choose a sequence $\{\gamma_n\} \subset C^{\tilde{q}-3}(\mathcal{V}|D_{j+1}, \mathcal{O})$ such that $j(r(\eta_n)) = \delta\gamma_n$. Then

$$\xi - \delta\mu - \delta\gamma_n = j(\eta - r(\eta_n))$$

This proves that $\delta\mu + \delta\gamma_n$ converges to ξ when $n \to \infty$. Since, by lemma 3.1, $dim_{\mathbb{C}} H^{\tilde{q}-2}(D_{j+1}, \mathcal{O}) < \infty$, then the coboundary space $B^{\tilde{q}-2}(\mathcal{V}|_{D_{j+1}}, \mathcal{O})$ is closed in $Z^{\tilde{q}-2}(\mathcal{V}|_{D_{j+1}}, \mathcal{O})$. Therefore $\xi \in B^{\tilde{q}-2}(\mathcal{V}|_{D_{j+1}}, \mathcal{O})$ and $H^{\tilde{q}-2}(D_j, \mathcal{O}) = 0$ for all $0 \leq j \leq k$.

Now since $D_\varepsilon \subset\subset D_k(\varepsilon) = D_k$, there exists $0 < \varepsilon' < \varepsilon$ such that $\varepsilon' > \varepsilon_0$ and $D_{\varepsilon'} \subset\subset D_k(\varepsilon)$. Then we have $D_\varepsilon \subset D_{\varepsilon'} \subset D_k(\varepsilon) \subset D_k(\varepsilon')$. Since for every $p \geq \tilde{q}-2$, $H^p(D_k(\varepsilon'), \mathcal{O}) \to H^p(D_{\varepsilon'}, \mathcal{O})$ is surjective by lemma 1, then $H^p(D_k(\varepsilon), \mathcal{O}) \to H^p(D_{\varepsilon'}, \mathcal{O})$ is also surjective for all $p \geq \tilde{q} - 2$, which shows that $H^p(D_{\varepsilon'}, \mathcal{O}) = 0$ for $p \geq \tilde{q} - 2$ and $\varepsilon' \in A$. ■

Proposition 3.4. The set D_{ε_o} is cohomologially $(\tilde{q} - 3)$-complete with respect to the structure sheaf $\mathcal{O}_{D_{\varepsilon_0}}$, which means that $H^p(D_{\varepsilon_0}, \mathcal{O}_{D_{\varepsilon_0}}) = 0$ for all $p \geq \tilde{q} - 3$.

Proof. In order to prove the Proposition, we consider the set A of all real numbers $\varepsilon \geq \varepsilon_o$ such that $H^p(D_\varepsilon, \mathcal{O}) = 0$ for all $p \geq \tilde{q} - 2$, where $D_\varepsilon = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon\}$. Then, by lemma 3, the set A is not empty and open in $[+\varepsilon_0, +\infty[$. Moreover, if $\varepsilon = Inf(A)$, there exists a decreasing sequence of real numbers $\varepsilon_j \in A$, $j \in \mathbb{N}$, such that $\varepsilon_j \to \varepsilon$. Since $H^p(D_{\varepsilon_j}, \mathcal{O}) = 0$ for all $p \geq \tilde{q} - 2$ and, by lemma 3.2, $H^p(D_{\varepsilon_{j+1}}, \mathcal{O}) \to H^p(D_{\varepsilon_j}, \mathcal{O})$ has a dense image for $p \geq \tilde{q} - 3$, then $H^p(D_\varepsilon, \mathcal{O}) = 0$ for all $p \geq \tilde{q} - 2$ (see [1], p. 250). Hence $\varepsilon \in A$.

Suppose now that $\varepsilon > \varepsilon_0$. then there exists $\varepsilon' \in A$ such that $\varepsilon_0 < \varepsilon' < \varepsilon$, which is a contradiction. Therefore $\varepsilon_0 = \varepsilon \in A$. But, since $H^{\tilde{q}-3}(D_{\varepsilon_0}, \mathcal{O}) = 0$ according to the proof of lemma 2, then D_{ε_0} is cohomologically $(\tilde{q} - 3)$-complete. ■

Theorem 3.5. Let (n, q) be a pair of integers such that $n \geq 3$, $1 < q < n$, and q does not divide n. We put $m = \left[\frac{n}{q}\right]$ and $r = n - mq$. Then there exists a cohomologically $(\tilde{q} - 3)$-complete domain D with respect to O_D in $\mathbb{C}^n$ such that $H_{n+\tilde{q}-3}(D, \mathbb{Z}) \neq 0$, if $r = 1$ and $m > q \geq 2$, where $\tilde{q} = n - \left[\frac{n}{q}\right] + 1$ and $\left[\frac{n}{q}\right]$ is the integral part of $\frac{n}{q}$.

Proof. We have proved in the Proposition that D_{ε_o} is cohomologically $(\tilde{q}-3)$-complete with respect to the structure sheaf $\mathcal{O}_{D_{\varepsilon_0}}$.

It was shown by Diederich-Fornaess [4] that if $\delta > 0$ is small enough, then the following topological sphere of real dimension $n + \tilde{q} - 2$:

$$S_\delta = \{z \in \mathbb{C}^n : \ x_1^2 + \cdots + x_m^2 + |z_{m+1}|^2 + \cdots + |z_n|^2 = \delta,$$

$$y_j = - \sum_{i=m+1}^{n} |z_i|^2 + (m+1) \left(\sum_{i=m+(j-1)(q-1)+1}^{m+j(q-1)} |z_i|^2 \right) \text{ for}$$

$$j = 1, \cdots, m\},$$

where $z_j = x_j + y_j$ for $j = 1, \cdots, n$, is not homologous to 0 in D_{ε_o}. This follows from the fact that the set $E = \{z \in \mathbb{C}^n : x_1 = \cdots = x_m = z_{m+1} = \cdots = z_n = 0\}$ does not intersect D_{ε_o}.

Let $0 < \varepsilon < \varepsilon_0$ be such that $D_\varepsilon \subset\subset B$. Then it follows, exactly as for D_{ε_o}, that D_ε is cohomologically $(\tilde{q} - 2)$-complete with respect to $\mathcal{O}_{D_\varepsilon}$. We may take ε such that D_ε does not intersect E. Then S_δ is not homologous to 0 in D_ε.

We are now going to prove that $H_{n+\tilde{q}-3}(D_\varepsilon, \mathbb{Z})$ does not vanish. Indeed, we define for $1 \leq j \leq m+1$ the sets $E_j = (D_1 \cap \cdots \cap D_{m-j+2}) \cup (D_{m-j+3} \cup \cdots \cup D_{m+1})$. Then by using the Mayer-Vietoris sequence for homology

$$\begin{aligned} 0 &= H_{n+\tilde{q}+j-3}(D_1 \cap \cdots \cap D_{m-j+2}, \mathbb{Z}) \oplus H_{n+\tilde{q}+j-3}(D_{m-j+3} \cup \cdots \cup D_{m+1}, \mathbb{Z}) \\ &\to H_{n+\tilde{q}+j-3}(E_{j+1}, \mathbb{Z}) \to H_{n+\tilde{q}+j-4}(E_j, \mathbb{Z}) \\ &\to H_{n+\tilde{q}+j-4}(D_1 \cap \cdots \cap D_{m-j+2}, \mathbb{Z}) \oplus H_{n+\tilde{q}+j-4}(D_{m-j+3} \cup \cdots \cup D_{m+1}, \mathbb{Z}) \to \cdots \end{aligned}$$

one can easily verify by induction that

$$H_{2n-2}(D_1 \cup \cdots \cup D_{m+1}, \mathbb{Z}) \to H_{n+\tilde{q}-3}(D_\varepsilon, \mathbb{Z})$$

is injective.

We first show that $H^{n+\tilde{q}-3}(D_\varepsilon, \mathbb{C}) \neq 0$. For this, we consider the $(2n-m-2)$-real form defined as follows:

$$\omega = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=m+2}^{n} y_i^2\right)^{-2n+m+1} \left(\sum_{i=1}^{n}(-1)^i x_i dx_1 \wedge \cdots \right.$$
$$\wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \wedge dy_{m+2} \wedge \cdots \wedge dy_{m+i} \wedge \cdots \wedge dy_n$$
$$+ \sum_{i=2}^{n-m}(-1)^{n+i-1} y_{m+i} dx_1 \wedge \cdots \wedge dx_n \wedge dy_{m+2} \wedge \cdots$$
$$\left. \wedge \widehat{dy_{m+i}} \wedge \cdots \wedge dy_n \right)$$

Then ω is d-closed and therefore defines a cohomology class in $H^{2n-m-2}(D_\varepsilon, \mathbb{C})$.

Let S'_δ be the topological sphere of real dimension $2n-m-2$ defined by $S'_\delta = \{z \in S_\delta : y_{m+1} = 0\}$, where $z_j = x_j + iy_j$ for $j = 1, \cdots, n$.

Since ω does not depend on $y_1, \cdots, y_{m+1}$, then

$$\int_{S'_\delta} \omega \neq 0$$

This implies that $H^{2n-m-2}(D_\varepsilon, \mathbb{C}) \neq 0$.

We shall prove now that if $E = D_1 \cup \cdots \cup D_{m+1}$, then the homology group $H_{2n-2}(E, \mathbb{Z})$ does not vanish. We note first that the map

$$H^{2n-m-2}(D_\varepsilon, \mathbb{C}) \to H^{2n-2}(E, \mathbb{C})$$

is an isomorphism. In fact, we consider over E the resolution of the constant sheaf $\mathbb{C}$:

$$0 \to \mathbb{C} \xrightarrow{i} \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0$$

where Ω^p is the sheaf of germs of holomorphic p-forms on E. This resolution maybe break it up into short exact sequences

$$0 \to Z^i \to \Omega^i \to Z^{i+1} \to 0, \quad for \ \ i = 0, \cdots, n$$

where $Z^0 = \mathbb{C}$, $\Omega^0 = \mathcal{O}$, and $Z^i = Im\{\Omega^{i-1} \xrightarrow{d} \Omega^i\}$ for $1 \leq i \leq n$.

On the other hand, since for any integer t with $1 \leq t \leq m-1$ and all $i_1, \cdots, i_t \in \{1, \cdots, m+1\}$, $D_{i_1} \cap \cdots \cap D_{i_t}$ is $((m-1)(q-1)+1)$-Runge in B and $n-m-2 \geq (m-1)(q-1)$, then $H^p(D_{i_1} \cap \cdots \cap D_{i_t}, \mathbb{C}) = 0$ for $p \geq 2n-m-2$. Hence for any indexes $j_1, \cdots, j_m \in \{1, \cdots, m+1\}$ we have

$$H^{2n-m-2}(D_{j_1} \cap \cdots \cap D_{j_m}, \mathbb{C}) \cong H^{2n-3}(D_{j_1} \cup \cdots \cup D_{j_m}, \mathbb{C})$$

But $H^{2n-3}(D_{j_1} \cup \cdots \cup D_{j_m}, \mathbb{C}) = 0$. Indeed, if we put $E_m = D_{j_1} \cup \cdots \cup D_{j_m}$, then obviously $H^p(E_m, \Omega^j) = 0$ for all $p \geq n-1$, and $j \geq 0$, because E_m is n-Runge in B. Note also that since $n-m-2 \geq (m-1)q-(m-1)$, then, by the proof of lemma 2, $H^{n-2}(E_m, \Omega^{n-1}) \cong H^{n-m-1}(D_{j_1} \cap \cdots \cap D_{j_m}, \Omega^{n-1}) = 0$ and $H^{n-3}(E_m, \Omega^n) \cong H^{n-m-2}(D_{j_1} \cap \cdots \cap D_{j_m}, \Omega^n) = 0$, because the Ω^j are free and $n-m-2 \geq q$ and $n-m-1 \leq n-q-2$. (See the proof of lemma 2). Therefore from the long exact sequences of cohomology associated to the short exact sequences

$$0 \to Z^i \to \Omega^i \to Z^{i+1} \to 0, \quad for \;\; i = 0, \cdots, n$$

we deduce a natural $\mathbb{C}$-isomorphism

$$H^{2n-3}(E_m, \mathbb{C}) \cong H^{n-2}(E_m, Z^{n-1})$$

and, an exact sequence

$$\cdots \to H^{n-3}(E_m, \Omega^n) \to H^{n-2}(E_m, Z^{n-1}) \to H^{n-2}(E_m, \Omega^{n-1}) \to \cdots$$

Since

$$H^{n-3}(E_m, \Omega^n) = H^{n-2}(E_m, \Omega^{n-1}) = 0,$$

then

$$H^{2n-3}(E_m, \mathbb{C}) \cong H^{n-2}(E_\varepsilon, Z^{n-1}) = 0,$$

which shows that $H^{2n-m-2}(D_\varepsilon, \mathbb{C}) \to H^{2n-2}(E, \mathbb{C})$ is an isomorphism.

Now by using the universal coefficient theorem for homology

$$H_{2n-2}(E, \mathbb{C}) \cong H_{2n-2}(E, \mathbb{Z}) \otimes \mathbb{C} \oplus Tor(H_{2n-3}(E, \mathbb{Z}), \mathbb{C})$$

and the fact that $Tor(H_{2n-3}(E, \mathbb{Z}), \mathbb{C}) = 0$, we conclude that $H_{2n-2}(E, \mathbb{Z}) \neq 0$. Since we know already that the map $H_{2n-2}(E, \mathbb{Z}) \to H_{n+\tilde{q}-3}(D_\varepsilon, \mathbb{Z})$ is injective, then $H_{n+\tilde{q}-3}(D_\varepsilon, \mathbb{Z})$ does not vanish. This completes the proof of the theorem. ■

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