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Quantum Algorithm for Perfect Matching Problem

Toru Fujimura

Chemical Department, Industrial Property Cooperation Center,
1-2-15, Kiba, Koto-ku, Tokyo 135-0042, Japan
E-mail: tfujimura8@gmail.com

Abstract

A quantum algorithm for the perfect matching problem and its example are reported. When each of two groups is consisted of n persons, it is decided whether there are n pairs or not. A computational complexity of a classical calculation is $n!$. The computational complexity becomes about $(n!)^{1/2}$ by the quantum algorithm that uses quantum phase inversion gates and quantum inversion about mean gates.

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1. Introduction

A quantum computer can move quickly to resolve a problem by doing a parallel calculation that uses quantum entangled states. Deutsch-Jozsa's algorithm for the rapid solution [1–3], Shor's algorithm for the factorization [2–4], Grover's algorithms for the database search [2,5,6], and so on are known. A quantum algorithm for the knapsack problem has recently been reported by Fujimura [7]. The perfect matching problem [8] is examined this time. Therefore, its result is reported.

2. Perfect Matching Problem

When each of two groups is consisted of n persons, it is decided whether there are n pairs or not [8].

3. Quantum Algorithm

It is assumed that there are the s-th person M_s [$0 \leq s \leq n-1$. s is an integer.] in one group and the t-th person F_t [$0 \leq t \leq n-1$. t is an integer.] in another group. When M_s selects F_t, $x_{s,t}$ is 1, or when M_s doesn't select F_t, $x_{s,t}$ is 0. When F_t selects M_s, $y_{t,s}$ is 1, or when F_t doesn't select M_s, $y_{t,s}$ is 0.

First of all, the quantum registers $|a_0>$, $|a_1>$, $\cdots$, $|a_{n-1}>$, and $|b>$ are prepared. When α is a minimum integer that is $log_2 n$ or more, each of $|a_f>$ that f is an integer from 0 to $n-1$ is consisted of α quantum bits [=qubits]. States of $|a_f>$ and $|b>$ are a_f and b, respectively.

Step 1: Each qubit of $|a_f>$ and $|b>$ is set $|0>$.

Step 2: The Hadamard gate $\boxed{\text{H}}$ [2, 3] acts on each qubit of $|a_f>$. It changes them for entangled states. The total states are $(2^\alpha)^n$.

Step 3: It is assumed that a quantum gate (A) changes $|b>$ for $|b+1>$ at $a_f \geq n$, or it doesn't change $|b>$ at $a_f < n$. As a target state for $|b>$ is 0, quantum phase inversion gates (PI) and quantum inversion about mean gates (IM) [2,5,6] act on $|b>$. When β is a minimum even integer that is $(2^\alpha/n)^{1/2}$ or more, the total number that (PI) and (IM) act on $|b>$ is β because they are a couple. Next, an observation gate (OB) observes $|b>$. These actions are repeated sequentially from $|a_0>$ to $|a_{n-1}>$. Therefore, each state of $|a_f>$ is 0, 1, $\cdots$, $n-2$, or $n-1$, and the total states become $n^n[=W_0]$.

Step 4: It is assumed that a quantum gate (B_0) changes $|b>$ for $|b+1>$ at $a_0 = 0$, or it doesn't change $|b>$ at $a_0 \neq 0$. It changes $|b>$ for $|b+1>$ at $a_1 = 0$, or it doesn't change $|b>$ at $a_1 \neq 0$. This action repeats to a_{n-1}. As a target state for $|b>$ is 1, (PI) and (IM) act on $|b>$. When γ_1 is a minimum even integer that is $(W_0/W_1)^{1/2}$ [$W_1 = n(n-1)^{n-1}$] or more, the total number that (PI) and (IM) act on $|b>$ is γ_1. By these actions, combinations that include only one 0 [W_1] are selected. Next, (OB) observes $|b>$, and combinations of W_1 remain. And $|b>$ is set $|0>$. Similarly, (B_u) [$1 \leq u \leq n-2$. u is an integer.] changes $|b>$ for $|b+1>$ at $a_0 = u$, or it doesn't change $|b>$ at $a_0 \neq u$. It changes $|b>$ for $|b+1>$ at $a_1 = u$, or it doesn't change $|b>$ at $a_1 \neq u$. This action repeats to a_{n-1}. As a target state for $|b>$ is 1, (PI) and (IM) act on $|b>$. When γ_{u+1} is a minimum even integer that is $(W_u/W_{u+1})^{1/2}$ [$W_{u+1} = n(n-1)\cdots(n-u)(n-(u+1))^{n-(u+1)}$] or more, the total number that (PI) and (IM) act on $|b>$ is γ_{u+1}. By these actions, combinations that include only one u [W_{u+1}] are selected. Next, (OB) observes $|b>$, and combinations of W_{u+1} remain. And $|b>$ is set $|0>$. These actions are repeated sequentially from 1 to $n-2$ at u. After all, combinations that include different numbers from 0 to $n-1$ [$W_{n-1} = n!$] remain.

Step 5: It is assumed that a quantum gate (C_0) obtains a state of $|a_0>$, and it changes $|b>$ for $|b + x_{0,a_0} y_{a_0,0}>$. Similarly, (C_w) [$1 \leq w \leq n-1$. w is an integer.] obtains a state of $|a_w>$, and it changes $|b>$ for $|b + x_{w,a_w} y_{a_w,w}>$. These actions are repeated sequentially from 1 to $n-1$ at w. As a target state for $|b>$ is n, (PI) and (IM) act on $|b>$. When γ_n is a minimum even integer that is $(W_{n-1})^{1/2}$ or more, the total number that (PI) and (IM) act on $|b>$ is γ_n. Next, (OB) observes $|a_0>, |a_1>, \cdots, |a_{n-1}>$, and $|b>$. By these actions, $a_0, a_1, \cdots, a_{n-1}$, and b are obtained. Therefore, when b is n, a perfect matching that is $(M_0, F_{a_0}), (M_1, F_{a_1}), \cdots, (M_{n-2}, F_{a_{n-2}})$, and $(M_{n-1}, F_{a_{n-1}})$ is obtained.

4. Numerical Calculation

It is assumed that there are the s-th person M_s $[0 \leq s \leq 4]$ in one group, the t-th person F_t $[0 \leq t \leq 4]$ in another group, $x_{0,1} = x_{0,2} = x_{0,3} = x_{1,0} = x_{1,4} = x_{2,0} = x_{2,1} = x_{2,3} = x_{3,0} = x_{3,2} = x_{3,4} = x_{4,0} = x_{4,2} = x_{4,3} = x_{4,4} = 1$, (others of $x_{s,t}$) $= 0$, $y_{0,0} = y_{0,3} = y_{1,0} = y_{1,1} = y_{1,2} = y_{2,1} = y_{2,2} = y_{2,4} = y_{3,0} = y_{3,1} = y_{3,3} = y_{3,4} = y_{4,1} = y_{4,2} = y_{4,3} = 1$, and (others of $y_{t,s}) = 0$.

First of all, $|a_0>, |a_1>, \cdots, |a_4>$, and $|b>$ are prepared. As $log_2 n$ is $log_2 5 \approx 2.3 \leq 3 = \alpha$, each of $|a_f>$ that f is an integer from 0 to 4 is consisted of 3 qubits.

Step 1: Each qubit of $|a_f>$ and $|b>$ is set $|0>$.

Step 2: $\boxed{\text{H}}$ acts on each qubit of $|a_f>$. It changes them for entangled states. The total states are $(2^\alpha)^n = (2^3)^5$.

Step 3: (A) changes $|b>$ for $|b+1>$ at $a_f \geq 5$, or it doesn't change $|b>$ at $a_f < 5$. As a target state for $|b>$ is 0, (PI) and (IM) act on $|b>$. When β is a minimum even integer that is $(2^\alpha/n)^{1/2} = (2^3/5)^{1/2} \approx 1.3 \leq 2 = \beta$, the total number that (PI) and (IM) act on $|b>$ is 2. Next, (OB) observes $|b>$. These actions are repeated sequentially from $|a_0>$ to $|a_4>$. Therefore, each state of $|a_f>$ is 0, 1, 2, 3, or 4, and the total states become $n^n = 5^5 [= W_0]$.

Step 4: (B_0) changes $|b>$ for $|b+1>$ at $a_0 = 0$, or it doesn't change $|b>$ at $a_0 \neq 0$. It changes $|b>$ for $|b+1>$ at $a_1 = 0$, or it doesn't change $|b>$ at $a_1 \neq 0$. This action repeats to a_4. As a target state for $|b>$ is 1, (PI) and (IM) act on $|b>$. When γ_1 is a minimum even integer that is $(W_0/W_1)^{1/2}$ $[W_1 = 5 \cdot 4^4]$ or more, the total number that (PI) and (IM) act on $|b>$ is γ_1. By these actions, combinations that include only one 0 $[W_1]$ are selected. Next, (OB) observes $|b>$, and combinations of W_1 remain. And $|b>$ is set $|0>$. Similarly, (B_u) $[1 \leq u \leq 3]$ changes $|b>$ for $|b+1>$ at $a_0 = u$, or it doesn't change $|b>$ at $a_0 \neq u$. It changes $|b>$ for $|b+1>$ at $a_1 = u$, or it doesn't change $|b>$ at $a_1 \neq u$. This action repeats to a_4. As a target state for $|b>$ is 1, (PI) and (IM) act on $|b>$. When γ_{u+1} is a minimum even integer that is $(W_u/W_{u+1})^{1/2}$ $[W_{u+1} = 5(5-1)\cdots(5-u)(5-(u+1))^{5-(u+1)}]$ or more, the total number that (PI) and (IM) act on $|b>$ is γ_{u+1}. By these actions, combinations that include only one u $[W_{u+1}]$ are selected. Next, (OB) observes $|b>$, and combinations of W_{u+1} remain. And $|b>$ is set $|0>$. These actions are repeated sequentially from 1 to 3 at u. Where, $(W_0/W_1)^{1/2}$, $(W_1/W_2)^{1/2}$, $(W_2/W_3)^{1/2}$, and $(W_3/W_4)^{1/2}$ are about 1.6, 1.5, 1.5, and 1.4, respectively. Therefore, $\gamma_1, \gamma_2, \gamma_3$, and γ_4 are same value 2. After all, combinations that include different numbers from 0 to 4 $[W_4 = 5!]$ remain.

Step 5: (C_0) obtains a state of $|a_0>$, and it changes $|b>$ for $|b+x_{0,a_0}y_{a_0,0}>$. Similarly, (C_w) $[1 \leq w \leq 4]$ obtains a state of $|a_w>$, and it changes $|b>$ for $|b+x_{w,a_w}y_{a_w,w}>$. These actions are repeated sequentially from 1 to 4 at w. As a target state for $|b>$ is $n = 5$, (PI) and (IM) act on $|b>$. When γ_5 is a minimum even integer that is $(W_4)^{1/2} = (5!)^{1/2} \approx 11.0 \leq 12 = \gamma_5$, the total number that (PI) and (IM) act on $|b>$ is $\gamma_5 = 12$. Next, (OB) observes $|a_0>, |a_1>, \cdots, |a_4>$, and $|b>$. For example, when a_0, a_1, a_2, a_3, a_4, and b are 3, 4, 1, 0, 2, and 5, respectively, there is a combination of a perfect matching that is (M_0, F_3), (M_1, F_4), (M_2, F_1), (M_3, F_0), and (M_4, F_2).

5. Discussion and Summary

The computational complexity of this quantum algorithm [$= S$] becomes the following. In the order of the actions by the gates, the number of them is αn at $\boxed{\text{H}}$, n at (A), $\beta n = 2n$ at (PI) and (IM), n at (OB), $n(n-1)$ at (B_d) $[0 \leq d \leq n-2.$ d is an integer.], $\sum_{i=1}^{n-1} \gamma_i$ at (PI) and (IM), $n-1$ at (OB), n at (C_e) $[0 \leq e \leq n-1.$ e is an integer.], γ_n at (PI) and (IM), and 1 at (OB). Therefore, S becomes $n^2 + (\alpha + 5)n + \sum_{i=1}^{n} \gamma_i$. In the example of the section 4, S is 85. The computational complexity of the classical calculation [$= Z$] is $n! = 5! = 120$. After all, S/Z becomes about 2/3. When n is large enough, S becomes about $\gamma_n \approx (n!)^{1/2}$, and S/Z is about $1/(n!)^{1/2}$. For example, as for $n = 50$, S/Z is about $1/(1.7 \times 10^{32})$.

Applications of the quantum computation are expanded by this algorithm.

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