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Quantum Algorithm for Vertex Coloring Problem

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Abstract

A quantum algorithm for the vertex coloring problem and its example are reported. When n vertexes are connected by m edges, and both vertexes of each edge are different colors, a number of colors that is k is decided. A computational complexity of a classical calculation is k^n. The computational complexity becomes about $k^{n/2}$ by the quantum algorithm that uses quantum phase inversion gates and quantum inversion about mean gates. Therefore, a high-speed process becomes possible.

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1. Introduction

Quantum algorithms that have been started by Deutsch-Jozsa's algorithm for the rapid solution [1–3] are expanded the application range by Shor's algorithm for the factorization [2–4], Grover's algorithms for the database search [2,5,6], and so on. A quantum algorithm for the knapsack problem has recently been reported by Fujimura [7]. It used quantum phase inversion gates and quantum inversion about mean gates that were suggested by Grover [2,5,6]. The vertex coloring problem [8] is examined this time to use these gates. Therefore, its result is reported.

2. Vertex Coloring Problem

When n vertexes are connected by m edges, and both vertexes of each edge are different colors, a number of colors that is k is decided [8].

3. Quantum Algorithm

It is assumed that n vertexes are connected by m edges, and two vertexes have only one edge, because several edges don't change the essence of this problem. Therefore, when there is an edge between the i-th vertex and the j-th vertex, $x_{i,j}$ [$0 \leq i < j \leq n-1$. i and j are integers.] is 1, and when there isn't an edge between these vertexes, it is 0. Now, it is assumed that a number of colors is k. First of all, the quantum registers $|a_0>$, $|a_1>$, $\cdots$, $|a_{n-1}>$, and $|b>$ are prepared. When α is a minimum integer that is log_2k or more, each of $|a_f>$ that f is an integer from 0 to $n-1$ is consisted of α quantum bits [=qubits]. States of $|a_f>$ and $|b>$ are a_f and b, respectively.

Step 1: Each qubit of $|a_f>$ and $|b>$ is set $|0>$.

Step 2: The Hadamard gate $\boxed{\text{H}}$ [2, 3] acts on each qubit of $|a_f>$. It changes them for entangled states. The total states are $(2^\alpha)^n$.

Step 3: It is assumed that a quantum gate (A) changes $|b>$ for $|b+1>$ at $a_f \geq k$, or it doesn't change $|b>$ at $a_f < k$. As a target state for $|b>$ is 0, quantum phase inversion gates (PI) and quantum inversion about mean gates (IM) act on $|b>$. When β is a minimum even integer that is $(2^\alpha/k)^{1/2}$ or more, the total number that (PI) and (IM) act on $|b>$ is β because they are a couple. Next, an observation gate (OB) observes $|b>$. These actions are repeated sequentially from $|a_0>$ to $|a_{n-1}>$. Therefore, each state of $|a_f>$ is $0, 1, \cdots, k-2$, or $k-1$, and the total states become $k^n[= W_0]$.

Step 4: It is assumed that a quantum gate $(B_{i,j})$ [$0 \leq i < j \leq n-1$. i and j are integers.] changes $|b>$ for $|b+x_{i,j}>$ at $a_i \neq a_j$, or it doesn't change $|b>$ at $a_i = a_j$. These actions are repeated for all combinations of i and j. Therefore, combinations that both vertexes of each edge are different colors are selected. A minimum number of the combinations is $k![= W_{min}]$ because a number of them is $k!$ or more. As a target state for $|b>$ is m, (PI) and (IM) act on $|b>$. When γ is a minimum even integer that is $(W_0/W_{min})^{1/2}$ or more, the total number that (PI) and (IM) act on $|b>$ is γ. Next, (OB) observes $|a_f>$ and $|b>$. By these actions, a_f and b are obtained. Therefore, one answer is obtained. From this answer, we can know at least $k!$ answers.

4. Numerical Calculation

It is assumed that $k = 3, n = 7, m = 7, x_{0,1} = x_{1,2} = x_{1,3} = x_{1,4} = x_{4,5} = x_{4,6} = x_{5,6} = 1$, and (others of $x_{i,j}$) $= 0$ [$0 \leq i < j \leq 6$]. First of all, $|a_0>$, $|a_1>$, $\cdots$, $|a_6>$, and $|b>$ are prepared. As log_2k is $log_23 \approx 1.6' \leq 2 = \alpha$, each of $|a_f>$ that f is an integer from 0 to 6 is consisted of 2 qubits.

Step 1: Each qubit of $|a_f>$ and $|b>$ is set $|0>$.

Step 2: $\boxed{\text{H}}$ acts on each qubit of $|a_f>$. It changes them for entangled states. The total states are $(2^\alpha)^n = (2^2)^7$.

Step 3: (A) changes $|b>$ for $|b+1>$ at $a_f \geq 3$, or it doesn't change $|b>$ at $a_f < 3$. As a target state for $|b>$ is 0, (PI) and (IM) act on $|b>$. When β is a minimum even integer that is $(2^\alpha/k)^{1/2} = (2^2/3)^{1/2} \approx 1.2 \leq 2 = \beta$, the total number that (PI) and (IM) act on $|b>$ is 2. Next, (OB) observes $|b>$. These actions are repeated sequentially from $|a_0>$ to $|a_6>$. Therefore, each state of $|a_f>$ is 0, 1, or 2, and the total states become $k^n = 3^7 [= W_0]$.

Step 4: $(B_{i,j})$ $[0 \leq i < j \leq 6]$ changes $|b>$ for $|b + x_{i,j}>$ at $a_i \neq a_j$, or it doesn't change $|b>$ at $a_i = a_j$. These actions are repeated for all combinations of i and j. Therefore, combinations that both vertexes of each edge are different colors are selected. A minimum number of the combinations is $k! = 3! [= W_{min}]$ because a number of them is 3! or more. As a target state for $|b>$ is $m = 7$, (PI) and (IM) act on $|b>$. When γ is a minimum even integer that is $(W_0/W_{min})^{1/2} = (3^7/3!)^{1/2} \approx 19.1 \leq 20 = \gamma$, the total number that (PI) and (IM) act on $|b>$ is $\gamma = 20$. Next, (OB) observes $|a_0>, |a_1>, \cdots, |a_6>$, and $|b>$. By these actions, for example, $a_0, a_1, \cdots, a_6$, and b are 0, 1, 0, 0, 0, 1, 2, and 7, respectively. Therefore, one answer is 0, 1, 0, 0, 0, 1, and 2. From this answer, we can know at least 3! answers.

5. Discussion and Summary

The computational complexity of this quantum algorithm $[= S]$ becomes the following. In the order of the actions by the gates, the number of them is αn at $\boxed{\mathrm{H}}$, n at (A), $\beta n = 2n$ at (PI) and (IM), n at (OB), $n(n-1)$ at $(B_{i,j})$ $[0 \leq i < j \leq n-1]$, γ at (PI) and (IM), and 1 at (OB). Therefore, S becomes $n^2 + (\alpha+3)n + \gamma + 1$. In the example of the section 4, S is 105. The computational complexity of the classical calculation $[= Z]$ is $k^n = 3^7 = 2187$. After all, S/Z becomes about 1/20. When n is large enough, S becomes about $\gamma \approx k^{n/2}$, and S/Z is about $1/k^{n/2}$. For example, as for $k = 3$ and $n = 100$, S/Z is about $1/(7.2' \times 10^{23})$. Therefore, a high-speed process becomes possible.

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