Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 7, Number 4 (2011), pp. 375–381

http://www.ripublication.com/gjpam.htm

Ad-Fourier-Stieltjes Transform of Vector Measures on Compact Lie Groups

Yaogan Mensah[1]
Department of Mathematics,
Faculty of Sciences, University of Lomé
E-mail: mensahyaogan2@yahoo.fr
International Chair in Mathematical Physics
and Applications (ICPMA-UNESCO Chair),
University of Abomey-Calavi, Rep. of Benin

Yaovi M. Awussi
International Chair in Mathematical Physics
and Applications (ICPMA-UNESCO Chair),
University of Abomey-Calavi, Rep. of Benin
E-mail: mensahyaogan2@yahoo.fr, jawussi@yahoo.fr

Abstract

Let G be a compact Lie group and $\mathfrak{h}_{Ad}$ the minimal subalgebra of the Lie algebra $\mathfrak{g}$ of G on which the adjoint representation is unitary and irreducible. We define a transformation of Fourier type of bounded vector measures with respect to the adjoint representation of G and obtain, among other results, a characterization of the topological dual of $\mathfrak{h}_{Ad} \otimes \overline{\mathfrak{h}_{Ad}}$.

AMS Subject Classification: 43A30, 43A77.
Keywords: Compact Lie group, vector measure, Banach algebra, adjoint representation, Fourier-Stieltjes transformation.

1. Introduction

Since its introduction in Mathematics the powerful method of Fourier transform plays a prominent rôle in the investigation of properties of real or complex valued functions.

[1] Corresponding Author.

It may be very interesting to investigate similar methods and results for vector valued functions and vector measures. That is why in [1] the authors introduced the Fourier-Stieltjes transform of vector measures on compact groups and proved the analogues of some classical properties of Fourier transformation. Some fundamental spaces of transformed measures appeared in their work and some of their topological properties (mainly duality relations) were studied in [6] and [7].

In this paper, we are interested in a transformation involving the adjoint representation of a compact Lie group G. More precisely, we investigate properties of the Fourier coefficient at the adjoint representation of G of Banach algebra valued measures.

The rest of the paper is organized as follows. In section 2, we recall what is a vector measure and complete the paper in section 3 by stating our main results.

2. Preliminaries

This section draws heavily from [1] and [4].

Let $\mathcal{S}$ be a locally compact Hausdorff space. We denote by $\mathcal{K}(\mathcal{S})$ the space of continuous complex valued functions on $\mathcal{S}$ with compact support. Let E be a Banach space. Any linear mapping $m : \mathcal{K}(\mathcal{S}) \to E$ is called a *vector measure* on $\mathcal{S}$ if for any compact set $K \subset \mathcal{S}$, there exists a real $a_K > 0$ such that if $f \in \mathcal{K}(\mathcal{S})$ and $supp(f) \subset K$ then

$$\|m(f)\| \leq a_K \sup\{|f(t)| : t \in K\}. \tag{2.1}$$

The value $m(f)$ is denoted $\int_S f dm$.

An equivalent approach of the definiton of vector measures can be found in [3]. Let us point out that if $\mathcal{S}$ is compact then a vector measure on $\mathcal{S}$ is just a linear mapping $m : \mathcal{K}(\mathcal{S}) \to E$ which is continuous in the uniform norm topology of $\mathcal{K}(\mathcal{S})$.

A vector measure m is said to be *dominated* if there exists a positive measure μ such that

$$\| \int_S f dm \| \leq \int_S |f| d\mu, \quad f \in \mathcal{K}(\mathcal{S}). \tag{2.2}$$

For a dominated vector measure m there exists a smallest positive measure $|m|$ called the *variation* or the *modulus* of m that dominates it. A positive measure is said to be *bounded* if it is continuous in the uniform norm topology of $\mathcal{K}(\mathcal{S})$ and a dominated vector measure is said to be bounded it is it dominated by a bounded positive measure. Therefore a dominated vector measure on a compact space is automatically bounded. For vector integration we refer to [3] or [4].

Throughout this paper, we consider a Banach algebra (which we denoted by A) valued vector measures over a compact Lie group G. The space of such measures will be denoted by $M^1(G, A)$. It is a Banach algebra when it is endowed with the following norm and convolution product:

$$\|m\| = \int_G \chi_G d|m|, \quad m * n(f) = \int \int_G f(st) dm(s) dn(t) \tag{2.3}$$

where $m, n \in M^1(G, A)$, $f \in \mathcal{C}(G)$, and $\mathcal{C}(G)$ and χ_G are the space of continuous complex functions on G and the characteristic function of G respectively.

3. Main Results

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. We denote by Ad the adjoint representation of G. We assume that there exists on $\mathfrak{g}$ an inner product $\langle\ ,\ \rangle$ which is Ad-invariant; that is

$$\langle Ad(g)X, Ad(g)Y\rangle = \langle X, Y\rangle,\ \forall g \in G,\ \forall X, Y \in \mathfrak{g}. \tag{3.1}$$

It is the case for instance when the Lie group G is a real compact connected group. The adjoint representation Ad is unitary with respect to this inner product. Let $\mathfrak{h}_{Ad}$ be a minimal Lie subalgebra of $\mathfrak{g}$ such that the representation Ad of G on $\mathfrak{h}_{Ad}$ is irreducible. In the sequel, we assume that $\mathfrak{h}_{Ad}$ is endowed with the above inner product. Since the group G is compact, the representation space $\mathfrak{h}_{Ad}$ is of finite dimension d_{Ad}. We fix once and for all an orthonormal basis $(X_1, \dots, X_{d_{Ad}})$ of $\mathfrak{h}_{Ad}$ as a canonical basis.

The following proposition is clearly obvious.

Proposition 3.1. Let A be a (complex) Banach algebra and m an A-valued measure on the Lie group G. The mapping

$$(X, Y) \mapsto \int_G \langle Ad(g)^\dagger X, Y\rangle dm(g), \tag{3.2}$$

where $Ad(g)^\dagger$ denotes the adjoint of the operator $Ad(g)$, is sesquilinear from $\mathfrak{h}_{Ad} \times \mathfrak{h}_{Ad}$ into A.

We state now our main definition.

Definition 3.2. We shall call *Ad-Fourier-Stieltjes transform* of a vector measure m the $A-$valued sesquilinear map $\mathcal{F}^{Ad}m$ on $\mathfrak{h}_{Ad} \times \mathfrak{h}_{Ad}$ defined by

$$\mathcal{F}^{Ad}m(X, Y) = \int_G \langle Ad(g)^\dagger X, Y\rangle dm(g), \quad X, Y \in \mathfrak{h}_{Ad}. \tag{3.3}$$

Let $S(\mathfrak{h}_{Ad} \times \mathfrak{h}_{Ad}, A)$ be the set of sesquilinear mappings from $\mathfrak{h}_{Ad} \times \mathfrak{h}_{Ad}$ into A. It is well known that $S(\mathfrak{h}_{Ad} \times \mathfrak{h}_{Ad}, A)$ is isomorphic to $\mathcal{B}(\mathfrak{h}_{Ad} \otimes \overline{\mathfrak{h}_{Ad}}, A)$, the space of bounded linear operators from $\mathfrak{h}_{Ad} \otimes \overline{\mathfrak{h}_{Ad}}$, equipped with the projective tensor product norm, into A, where $\overline{\mathfrak{h}_{Ad}}$ denotes the conjugate vector space to $\mathfrak{h}_{Ad}$. It is then possible to linearise the transformed measure $\mathcal{F}^{Ad}m$ by considering it as an element of the space $\mathcal{B}(\mathfrak{h}_{Ad} \otimes \overline{\mathfrak{h}_{Ad}}, A)$ defined by

$$\mathcal{F}^{Ad}m(X \otimes Y) = \int_G \langle Ad(g)^\dagger X, Y\rangle dm(g), \quad X \in \mathfrak{h}_{Ad},\ Y \in \overline{\mathfrak{h}_{Ad}}. \tag{3.4}$$

Let us denote by dg the normalized Haar measure of G. If $f \in L_1(G, dg, A)$ then the Ad-Fourier transform of f is the Ad-Fourier-Stieltjes transform of the vector measure fdg, that is

$$\mathcal{F}^{Ad} f(X \otimes Y) = \int_G \langle Ad(g)^\dagger X, Y\rangle f(g)dg, \quad X \in \mathfrak{h}_{Ad},\ Y \in \overline{\mathfrak{h}_{Ad}}. \tag{3.5}$$

This formula remains valid for complex valued functions with the remark that the result $\mathcal{F}^{Ad} f(X \otimes Y)$ is a complex number. See for instance in [5] for informations on Fourier transform of complex functions on compact groups.

Definition 3.3. The complex valued functions $u_{ij}^{Ad} : g \mapsto \langle Ad(g)X_j, X_i\rangle$ defined on G will be called *Ad-coefficients*.

The following theorem expresses the Ad-Fourier-Stieltjes transform in terms of Ad-coefficients.

Theorem 3.4. If $m \in M^1(G, A)$ then there exists a family $(a_{ij})_{1\le i,j\le d_{Ad}}$ of elements of A such that

$$\mathcal{F}^{Ad} m = d_{Ad} \sum_{i,j=1}^{d_{Ad}} a_{ij} \mathcal{F}^{Ad} u_{ij}^{Ad}. \tag{3.6}$$

Proof. Let $X \in \mathfrak{h}_{Ad}$, $Y \in \overline{\mathfrak{h}_{Ad}}$. We express X and Y in the basis $(X_1, \dots, X_{d_{Ad}})$:

$$X = \sum_k \beta_k X_k,\ Y = \sum_l \gamma_l X_l.$$

Let $m \in M^1(G, A)$. One has

$$\mathcal{F}^{Ad} m(X \otimes Y) = \sum_k \sum_l \beta_k \overline{\gamma_l} \mathcal{F}^{Ad} m(X_k \otimes X_l). \tag{3.7}$$

In the other hand we have:

$$\begin{aligned}
\mathcal{F}^{Ad} u_{ij}^{Ad}(X \otimes Y) &= \int_G \langle Ad(g)^\dagger X, Y\rangle \langle Ad(g)X_i, X_j\rangle \\
&= \sum_k \sum_l \beta_k \overline{\gamma_l} \int_G \overline{\langle Ad(g)X_l, X_k\rangle} \langle Ad(g)X_i, X_j\rangle dg \\
&= \beta_j \overline{\gamma_i} \frac{1}{d_{Ad}}
\end{aligned}$$

according to Schur's orthogonality relations. See [5] or [2].

So $d_{Ad}\mathcal{F}^{Ad} u_{ij}^{Ad}(X \otimes Y) = \beta_j \overline{\gamma_i}$. Then we have

$$\mathcal{F}^{Ad} m(X \otimes Y) = \sum_j \sum_i \beta_j \overline{\gamma_i} \mathcal{F}^{Ad} m(X_j \otimes X_i) = d_{Ad} \sum_j \sum_i a_{ij} \mathcal{F}^{Ad} u_{ij}^{Ad}(X \otimes Y)$$

where we have put $a_{ij} = \mathcal{F}^{Ad}m(X_j \otimes X_i)$. ■

Let $\mathfrak{T}^{Ad}(G)$ denote the subspace of the space of continuous complex valued functions on G, $\mathcal{C}(G)$, spanned by the Ad-coefficients. We set

$$\mathfrak{T}^{Ad}(G, A) = \{a\varphi \,:\, a \in A, \varphi \in \mathfrak{T}^{Ad}(G)\}. \tag{3.8}$$

Theorem 3.5. The following equality holds:

$$\mathcal{F}^{Ad}(\mathfrak{T}^{Ad}(G, A)) = \mathcal{B}(\mathfrak{h}_{Ad} \otimes \overline{\mathfrak{h}_{Ad}}, A). \tag{3.9}$$

Proof. Let $\varphi \in \mathcal{B}(\mathfrak{h}_{Ad} \otimes \overline{\mathfrak{h}_{Ad}}, A)$. Put $a_{ij} = \varphi(X_j \otimes X_i)$ and $\psi = d_{Ad} \sum_{i,j=1}^{d_{Ad}} a_{ij}\mathcal{F}^{Ad}u_{ij}^{Ad}$.

We can show that $\varphi = \psi$.

$$\begin{aligned}
\psi(X_k \otimes X_l) &= d_{Ad} \sum_{i,j=1}^{d_{Ad}} a_{ij}\mathcal{F}^{Ad}u_{ij}^{Ad}(X_k \otimes X_l) \\
&= d_{Ad} \sum_{i,j=1}^{d_{Ad}} a_{ij} \int_G \langle Ad(g)^\dagger X_k, X_l\rangle\langle Ad(g)X_i, X_j\rangle dg \\
&= d_{Ad}a_{lk}\frac{1}{d_{Ad}} = a_{lk} = \varphi(X_k \otimes X_l).
\end{aligned}$$

So $\varphi = \psi$.

The converse is clearly obvious from Theorem 3.4. ■

If we take $A = \mathbb{C}$ we can derive from the above theorem the following characterization of $\left(\mathfrak{h}_{Ad} \otimes \overline{\mathfrak{h}_{Ad}}\right)^*$, the continuous dual of $(\mathfrak{h}_{Ad} \otimes \overline{\mathfrak{h}_{Ad}})$, with an integral representation.

Corollary 3.6. $\varphi \in \left(\mathfrak{h}_{Ad} \otimes \overline{\mathfrak{h}_{Ad}}\right)^*$ if and only if there exists a family (u_{ij}^{Ad}) of Ad-coefficients and a related family (λ_{ij}) of complex numbers such that $\sum_{ij} \lambda_{ij}\mathcal{F}^{Ad}u_{ij}^{Ad} = \varphi$, that is

$$\varphi(X \otimes Y) = \sum_{ij} \lambda_{ij} \int_G \langle Ad(g)^\dagger X, Y\rangle\langle Ad(g)X_i, X_j\rangle dg. \tag{3.10}$$

Now we associate with the Ad-Fourier-Stieltjes transform $\mathcal{F}^{Ad}m$ of a bounded vector measure m, the square matrix $(m_{ij})_{1\le i,j\le n}$ with entries in the Banach algebra A given by

$$m_{ij} = \mathcal{F}^{Ad}m(X_j \otimes X_i) \tag{3.11}$$

and we define a product $\times$ as follows: if $(m_{ij})_{1\le i,j\le d_{Ad}}$ and $(n_{ij})_{1\le i,j\le d_{Ad}}$ are associated with $\mathcal{F}^{Ad}m$ and $\mathcal{F}^{Ad}n$ respectively then $\mathcal{F}^{Ad}m \times \mathcal{F}^{Ad}n$ is the element of $\mathcal{B}(\mathfrak{h}_{Ad} \otimes \overline{\mathfrak{h}_{Ad}}, A)$ which matrix is $(n_{ij}).(m_{ij})$, the matrix product of $(n_{ij})_{1\le i,j\le d_{Ad}}$ and $(m_{ij})_{1\le i,j\le d_{Ad}}$.

We can now state the following theorem.

Theorem 3.7. The following equality holds:

$$\mathcal{F}^{Ad}(m * n) = \mathcal{F}^{Ad}m \times \mathcal{F}^{Ad}n. \tag{3.12}$$

Proof.

$$\begin{aligned} \mathcal{F}^{Ad}(m * n)(X_j \otimes X_i) &= \int_G \langle Ad(t)^{\dagger} X_j, Xi\rangle \, d(m * n)(t) \\ &= \int \int_G \langle Ad(st)^{\dagger} X_j, Xi\rangle dm(s)dn(t) \\ &= \int \int_G \langle Ad(t)^{\dagger} Ad(s)^{\dagger} X_j, Xi\rangle dm(s)dn(t). \end{aligned}$$

Now, we express $Ad(s)^{\dagger} X_j$ in the canonical basis of $\mathfrak{h}_{Ad}$:

$$Ad(s)^{\dagger} X_j = \sum_{k=1}^{n} \langle Ad(s)^{\dagger} X_j, X_k\rangle X_k. \tag{3.13}$$

So

$$\begin{aligned} \mathcal{F}^{Ad}(m * n)(X_j \otimes X_i) &= \sum_{k=1}^{n} \int \int_G \langle Ad(s)^{\dagger} X_j, X_k\rangle \langle Ad(t)^{\dagger} X_k, X_i\rangle dm(s)dn(t) \\ &= \sum_{k=1}^{n} \int_G \langle Ad(t)^{\dagger} X_k, X_i\rangle dn(t) \int_G \langle Ad(s)^{\dagger} X_j, X_k\rangle dm(s) \\ &= \sum_{k=1}^{n} \mathcal{F}^{Ad} n(X_k \otimes X_i) \mathcal{F}^{Ad} m(X_j \otimes X_k). \square \end{aligned}$$

■

References

[1] V.S.K. Assiamoua, A. Olubummo, Fourier-Stieltjes transforms of vector-valued measures on compact groups, *Acta Sci. Math.*, 53:301–307, 1989.

[2] A. O. Barut, R. Aczka, Theory of Group representations and Applications, PWN-Polish Scientific Publishers Warszawa, 1980.

[3] J. Diestel, J. Uhl, Vector measures, Math. Surveys 15, Amer. Math. Soc., Providence, 1977.

[4] N. Dinculeanu, Integration on Locally Compact Spaces, Noorhoff International Publishing, Leyden 1974.

[5] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis, Vol II, Springer-Verlag, New York-Berlin-Heidelberg, 1970.

[6] Y. Mensah, V. S. K. Assiamoua, The dual of the Fourier algebra $\mathcal{A}_1(G, A)$ of vector valued functions on compact groups, *Afr. Diaspora J. Math.*, 8(1):28–34, 2009.

[7] Y. Mensah, V. S. K. Assiamoua, On Spaces of Fourier-Stieltjes transform of vector measures on compact groups, *Mathematical Sciences*, 4(1):1–8, 2010.