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Normal or Hyponormal Weighted Composition Operators on the Fock-type Space

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Abstract

Weighted composition operators have been related to products of composition operators and their adjoints and to isometries of Hardy spaces. In this paper we obtain several results of normal or hyponormal weighted composition operators.

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1. Introduction

For two points $z = (z_1, \dots, z_N)$ and $w = (w_1, \dots, w_N)$ in $\mathbb{C}^N$, let $\langle z, w\rangle = \sum_{k=1}^{N} z_k \overline{w}_k$ and $|z| = \sqrt{\langle z, z\rangle}$. Let $dV(z)$ the Lebesgue volume measure on $\mathbb{C}^N$ and $H(\mathbb{C}^N)$ the space of all holomorphic functions on $\mathbb{C}^N$ (entire functions). For $\alpha > 0$ the Fock-type space $\mathscr{F}_\alpha^2(\mathbb{C}^N)$ is the space of all entire functions f on $\mathbb{C}^N$ for which

$$\|f\|^2 = \left(\frac{\alpha}{\pi}\right)^N \int_{\mathbb{C}^N} |f(z)|^2 e^{-\alpha|z|^2} dV(z) < \infty.$$

It is clear that $\mathscr{F}_\alpha^2(\mathbb{C}^N)$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \left(\frac{\alpha}{\pi}\right)^N \int_{\mathbb{C}^N} f(z)\overline{g(z)}e^{-\alpha|z|^2}dV(z),$$

and the reproducing kernel function is $K_w(z) = e^{\alpha\langle z,w\rangle}$.

Let $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ be an entire mapping and $\psi \in H(\mathbb{C}^N)$. The weighted composition operator $W_{\varphi,\psi}$ is defined by $W_{\varphi,\psi} f = \psi \cdot f \circ \varphi$. This operator has usually arisen answering other questions related to operators on spaces of holomorphic functions, such as questions about multiplication operators or composition operators. For example, weighted composition operators arise in the characterization of commutators of analytic Toeplitz operators (see [2]) and in the adjoints of composition operators (see [3]). Forelli [5] proved that the only isometry of Hardy space H^p for $p \neq 2$ is weighted composition operator. Recently, weighted composition operators have been discussed in many papers(see, e.g., [6]- [8], [9]). Carswell et al. [1] have determined when composition operators are bounded and compact on Fock space $\mathscr{F}_{\frac{1}{2}}^2(\mathbb{C}^N)$, and they have obtained the following result.

Theorem 1.1. Let $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ be an entire mapping.

(a) If the operator C_φ is bounded on $\mathscr{F}^2(\mathbb{C}^N)$, then $\varphi(z) = Az + b$, where A is an $N \times N$ matrix and b is an $N \times 1$ vector.

(b) If the operator C_φ is compact on $\mathscr{F}^2(\mathbb{C}^N)$, then $\varphi(z) = Az + b$, where $\|A\| < 1$.

Ueki [10] has given some necessary and sufficient conditions for weighted composition operators on Fock-type space $\mathscr{F}_\alpha^2(\mathbb{C})$ to be bounded and compact. Quite recently, Du [4] has obtained a complete description of Schatten class weighted composition operators on $\mathscr{F}_\alpha^2(\mathbb{C}^N)$.

In this short paper, we will obtain several properties of normal weighted composition operators and hyponormal weighted composition operators.

2. Main results

We first prove several auxiliary lemmas.

Lemma 2.1. Let $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ be an entire mapping and $\psi \in H(\mathbb{C}^N)$ and $W_{\varphi,\psi}$ be bounded on $\mathscr{F}_\alpha^2(\mathbb{C}^N)$. Then $W_{\varphi,\psi}^* K_w = \overline{\psi(w)} K_{\varphi(w)}$.

Proof. For each $z \in \mathbb{C}^N$, we have

$$\begin{aligned} W_{\varphi,\psi}^* K_w(z) &= \langle W_{\varphi,\psi}^* K_w, K_z\rangle = \langle K_w, W_{\varphi,\psi} K_z\rangle \\ &= \overline{\langle W_{\varphi,\psi} K_z, K_w\rangle} = \overline{\psi(w) K_z(\varphi(w))} \\ &= \overline{\psi(w)} K_{\varphi(w)}(z). \end{aligned}$$

This completes the proof. ■

Lemma 2.2. Let $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ be an entire mapping and $\psi \in H(\mathbb{C}^N)$. Then the bounded operator $W_{\varphi,\psi} : \mathscr{F}^2_\alpha(\mathbb{C}^N) \to \mathscr{F}^2_\alpha(\mathbb{C}^N)$ is Hermitian if and only if $W_{\varphi,\psi} K_w = W^*_{\varphi,\psi} K_w$ for all $w \in \mathbb{C}^N$.

Proof. If $W_{\varphi,\psi}$ is Hermitian, that is, $W_{\varphi,\psi} = W^*_{\varphi,\psi}$, then $W_{\varphi,\psi} K_w = W^*_{\varphi,\psi} K_w$. Conversely, for $w \in \mathbb{C}^N$ and $f \in \mathscr{F}^2_\alpha(\mathbb{C}^N)$, we have

$$\begin{aligned} W_{\varphi,\psi} f(w) &= \langle W_{\varphi,\psi} f, K_w\rangle = \langle f, W^*_{\varphi,\psi} K_w\rangle \\ &= \langle f, W_{\varphi,\psi} K_w\rangle = \langle W^*_{\varphi,\psi} f, K_w\rangle \\ &= W^*_{\varphi,\psi} f(w). \end{aligned}$$

It follows that $W_{\varphi,\psi} f = W^*_{\varphi,\psi} f$ for each $f \in \mathscr{F}^2_\alpha(\mathbb{C}^N)$, and then $W_{\varphi,\psi} = W^*_{\varphi,\psi}$. ■

Proposition 2.3. If the bounded operator $W_{\varphi,\psi} : \mathscr{F}^2_\alpha(\mathbb{C}^N) \to \mathscr{F}^2_\alpha(\mathbb{C}^N)$ is normal, then either $\psi \equiv 0$ or ψ never vanishes on $\mathbb{C}^N$.

Proof. Suppose $W_{\varphi,\psi}$ is normal and $\psi(z_0) = 0$ for some z_0 in $\mathbb{C}^N$. Then by Lemma 2.1, $W^*_{\varphi,\psi} K_{z_0} = \overline{\psi(w)} K_{\varphi(z_0)} \equiv 0$. Since $W_{\varphi,\psi}$ is normal, $\|W_{\varphi,\psi} K_{z_0}\| = \|W^*_{\varphi,\psi} K_{z_0}\|$. Therefore, $\|W_{\varphi,\psi} K_{z_0}\| = 0$ and thus $\psi(z) e^{\alpha\langle \varphi(z), z_0\rangle} = 0$ for each z in $\mathbb{C}^N$, which implies $\psi \equiv 0$. Thus, if $W_{\varphi,\psi}$ is normal, then either $\psi \equiv 0$ or ψ nonzero at every point in $\mathbb{C}^N$. ■

Proposition 2.4. Assume that the bounded operator $W_{\varphi,\psi} : \mathscr{F}^2_\alpha(\mathbb{C}^N) \to \mathscr{F}^2_\alpha(\mathbb{C}^N)$ is normal. If φ is not a constant and ψ is not the zero function, then φ is univalent.

Proof. Assume φ is not a constant and ψ is not the zero function. Then there exists points a and b such that $a \neq b$ and $\varphi(a) = \varphi(b)$. Since ψ is not the zero function, from Proposition 2.3, we get $\psi(a) \neq 0$ and $\psi(b) \neq 0$. Define

$$f = \frac{K_a}{\overline{\psi(a)}} - \frac{K_b}{\overline{\psi(b)}}$$

and observe f is a nonzero function in $\mathscr{F}^2_\alpha$. We have $W^*_{\varphi,\psi} f = 0$ by Lemma 2.1. Since $W_{\varphi,\psi}$ is normal, $\|W_{\varphi,\psi} f\| = \|W^*_{\varphi,\psi} f\| = 0$. But $\|W_{\varphi,\psi} f\| = 0$ implies $f \circ \varphi$ is the zero function. Since φ is nonconstant, f must vanish on a nonempty open subset of $\mathbb{C}^N$ and hence f must be the zero function, a contradiction. Hence φ is univalent. ■

If $W_{\varphi,\psi}$ is normal and $\varphi(a) = a$ for some a in $\mathbb{C}^N$, then ψ has the following form.

Proposition 2.5. Assume that the bounded operator $W_{\varphi,\psi} : \mathscr{F}^2_\alpha(\mathbb{C}^N) \to \mathscr{F}^2_\alpha(\mathbb{C}^N)$ is normal and $\varphi(a) = a$ for some a in $\mathbb{C}^N$. Then

$$\psi = \psi(a) \frac{K_a}{K_a \circ \varphi}.$$

Proof. By Lemma 2.1, we have $W^*_{\varphi,\psi}K_a = \overline{\psi(a)}K_a$, so that K_a is an eigenvector for $W^*_{\varphi,\psi}$ with corresponding eigenvalue $\overline{\psi(a)}$. Since $W_{\varphi,\psi}$ is normal, K_a is an eigenvector for $W_{\varphi,\psi}$ with corresponding eigenvalue $\psi(a)$ and thus it follows that

$$\psi(a)K_a = W_{\varphi,\psi}K_a = \psi K_a \circ \varphi,$$

from which the proposition holds. ■

Because $K_0 \equiv 1$, the preceding proposition and Theorem A yield the following corollary.

Corollary 2.6. Assume that $\varphi(0) = 0$. Then the bounded operator $W_{\varphi,\psi} : \mathscr{F}^2_\alpha(\mathbb{C}^N) \to \mathscr{F}^2_\alpha(\mathbb{C}^N)$ is normal if and only if ψ is constant and $\varphi(z) = Az$ for some normal $N \times N$ matrix A.

Proof. From the preceding proposition, we see that if $\varphi(0) = 0$, then $W_{\varphi,\psi}$ is normal if and only if ψ is constant and C_φ is normal. Thus we only prove that the normality of C_φ is equivalent to that of A. This is easy seen by $C^*_\varphi = C_\sigma$ where $\sigma(z) = A^*z$, when $\varphi(z) = Az$. ■

Corollary 2.6 motivates us to consider the problem: "What does it happen if φ has many fixed points in $\mathbb{C}^N$?" Indeed, if $\varphi(z) = Az + b$ and the rank of the matrix $(A - I, b) < N$, then φ has many fixed points. In order to identify this case, we need a lemma as follows.

Lemma 2.7. Let A be an $N \times N$ matrix, b be an $N \times 1$ vector and $C_\varphi : \mathscr{F}^2_\alpha(\mathbb{C}^N) \to \mathscr{F}^2_\alpha(\mathbb{C}^N)$ be bounded. Then $C^*_\varphi = W_{\sigma,g}$, where $\sigma(z) = A^*z$ and $g(z) = e^{\alpha\langle z,b\rangle}$.

Proof. In fact, by a similar proof, Theorem A is also true for $\mathscr{F}^2_\alpha(\mathbb{C}^N)$ and thus $\varphi(z) = Az + b$. For z and w in $\mathbb{C}^N$,

$$\begin{aligned} C^*_\varphi K_w(z) &= K_{\varphi(w)}(z) = e^{\alpha\langle z,\varphi(w)\rangle} = e^{\alpha\langle z,Aw+b\rangle} \\ &= e^{\alpha\langle A^*z,w\rangle}e^{\alpha\langle z,b\rangle} = W_{\sigma,g}K_w(z). \end{aligned}$$

In other words, $C^*_\varphi = W_{\sigma,g}$ on reproducing kernel functions. Since the span of these functions is dense in $\mathscr{F}^2_\alpha(\mathbb{C}^N)$, we have $C^*_\varphi = W_{\sigma,g}$. ■

Remark 2.8. By combing Lemma 2.2 and Lemma 2.7, we can also prove that if $\varphi(z) = Az + b$, then the bounded operator C_φ is Hermitian if and only if A is a Hermitian matrix and $b = 0$.

Theorem 2.9. Assume that $\varphi(z) = Az + b$ and φ has many fixed points in $\mathbb{C}^N$. Then the bounded operator $W_{\varphi,\psi} : \mathscr{F}^2_\alpha(\mathbb{C}^N) \to \mathscr{F}^2_\alpha(\mathbb{C}^N)$ is normal if and only if ψ is constant and $\varphi(z) = Az$ for some normal $N \times N$ matrix A.

Proof. Suppose $W_{\varphi,\psi}$ is normal on $\mathscr{F}^2_\alpha(\mathbb{C}^N)$. Let $\varphi(z) = Az + b$ and φ have many fixed points in $\mathbb{C}^N$. Then we see that there are many points z in $\mathbb{C}^N$ satisfying the linear equation system $\varphi(z) = Az + b = z$. We can choose a sequence $\{z_n\}_{n\in\mathbb{N}}$ of the fixed points of φ with $z_n \to z_0$ as $n \to \infty$. From proposition 2.5, we have

$$\psi = \psi(z_n)\frac{K_{z_n}}{K_{z_n} \circ \varphi},$$

from which it follows that $\psi(z_m) = \psi(z_n)$ for every $m, n \in \mathbb{N}$. By this and $z_n \to z_0$ as $n \to \infty$, ψ is a constant and thus $W_{\varphi,\psi} = aC_\varphi$. Because $W_{\varphi,\psi} = aC_\varphi$ is bounded on $\mathscr{F}^2_\alpha(\mathbb{C}^N)$, similar to the proof of Theorem A, we also have that $\varphi(z) = Az+b$. Therefore, we only need to prove that if $\varphi(z) = Az + b$ and C_φ is normal, then $\varphi(z) = Az$ and A is an $N \times N$ normal matrix.

By Lemma 2.7, we have

$$C_\varphi C^*_\varphi K_w(z) = C_\varphi W_{\sigma,g} K_w(z) = g \circ \varphi(z) K_w \circ \sigma \circ \varphi(z)$$

and

$$C^*_\varphi C_\varphi K_w(z) = W_{\sigma,g} C_\varphi K_w(z) = g(z) K_w \circ \varphi \circ \sigma(z).$$

From this and since C_φ is normal, it follows that

$$g \circ \varphi(z) K_w \circ \sigma \circ \varphi(z) = g(z) K_w \circ \varphi \circ \sigma(z). \tag{2.1}$$

Replacing g and σ by $g(z) = e^{\alpha\langle z,b\rangle}$ and $\sigma(z) = A^*z$ in (2.1), we have

$$\langle Az, b\rangle + |b|^2 + \langle Az, Aw\rangle + \langle b, Aw\rangle = \langle z, b\rangle + \langle A^*z, A^*w\rangle + \langle z, w\rangle + \frac{2\pi i}{\alpha}k(z, w), \tag{2.2}$$

$k(z, w) \in \mathbb{N}$. Setting $z = w = 0$ in (2.2), we see that $b = 0$ and $k(0, 0) = 0$. Since $\varphi(z) = Az$ and by Lemma 2.7, A must be a normal matrix.

Conversely, if ψ is constant and $\varphi(z) = Az$ for some $N \times N$ normal matrix A, then it is clear that $W_{\varphi,\psi}$ is normal. ■

Similar to Theorem 2.9, we have the following conclusion.

Theorem 2.10. Assume that $\varphi(z) = Az + b$ and φ has many fixed points in $\mathbb{C}^N$. Then the bounded operator $W_{\varphi,\psi} : \mathscr{F}^2_\alpha(\mathbb{C}^N) \to \mathscr{F}^2_\alpha(\mathbb{C}^N)$ is unitary if and only if $|\psi| \equiv 1$ and A is unitary and $b = 0$.

In the final of this section we supply some properties of hyponormal weighted composition operators. Recall that the bounded linear operator T on Hilbert space H is called *hyponormal* if $T^*T \geq TT^*$ where $\geq$ denotes the usual ordering on self-adjoint operators. Therefore this definition is easily seen to be equivalent to $\|Tx\| \geq \|T^*x\|$ for all vectors x in H.

Proposition 2.11. Assume that the bounded operator $W^*_{\varphi,\psi} : \mathscr{F}^2_\alpha(\mathbb{C}^N) \to \mathscr{F}^2_\alpha(\mathbb{C}^N)$ is hyponormal and $\varphi(a) = a$ for some point a in $\mathbb{C}^N$. Then

$$\psi = \psi(0)\frac{K_a}{K_a \circ \varphi}.$$

Proof. Suppose $\varphi(a) = a$ for some a in $\mathbb{C}^N$. By $W^*_{\varphi,\psi}K_a = \overline{\psi(a)}K_{\varphi(a)}$, we see that K_a is an eigenvector for $W^*_{\varphi,\psi}$. The hyponormality of $W^*_{\varphi,\psi}$ implies that K_a is also an eigenvector for $W_{\varphi,\psi}$. Then there is a number λ such that for all z,

$$\psi(z)K_a \circ \varphi(z) = W_{\varphi,\psi}K_a(z) = \lambda K_a(z). \tag{2.3}$$

Letting $z = 0$ in (2.3) shows $\lambda = \psi(0)$, and then the desired form of ψ follows. ■

For every bounded composition operator on $\mathscr{F}^2_\alpha(\mathbb{C}^N)$, we can obtain the result of hyponormality as follows.

Theorem 2.12. Assume that $C_\varphi : \mathscr{F}^2_\alpha(\mathbb{C}^N) \to \mathscr{F}^2_\alpha(\mathbb{C}^N)$ is bounded. Then C_φ is hyponormal if and only if $\varphi(z) = Az$ and A^* is hyponormal.

Proof. Fist suppose C_φ is hyponormal. The boundedness of C_φ implies that $\varphi(z) = Az + b$. Obviously, 1 is an eigenvector for C_φ. This shows that $K_0 = 1$ is an eigenvector for C^*_φ. Since $C^*_\varphi K_0 = K_{\varphi(0)}$, we get $b = 0$ which means $\varphi(z) = Az$. Noting that $\|C_\varphi K_w\|^2 = e^{\alpha|A^*w|^2}$ and $\|C^*_\varphi K_w\|^2 = e^{\alpha|Aw|^2}$, by the hyponormality of C_φ we have $|A^*w|^2 \geq |Aw|^2$ for all w. This means A^* is a hyponormal matrix.

Conversely, if A^* is is hyponormal, then it follows that

$$\|C_\varphi K_w\|^2 = e^{\alpha|A^*w|^2} \geq e^{\alpha|Aw|^2} = \|C^*_\varphi K_w\|^2$$

for all w. In other words, $\|C_\varphi K_w\| \geq \|C^*_\varphi K_w\|$ on reproducing kernel functions. Since the span of these functions is dense in $\mathscr{F}^2_\alpha(\mathbb{C}^N)$, we have $\|C_\varphi f\| \geq \|C^*_\varphi f\|$ for all f in $\mathscr{F}^2_\alpha(\mathbb{C}^N)$, which shows C_φ is hyponormal. ■

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