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Radius Problem for a Class of Convex Functions

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Abstract

In this paper we study some radius problems of function of the type $\frac{1}{w}f(wz)$, where f belongs to a subclass of the class of convex normalized and univalent functions on the unit disc D. A class $\mathcal{D}(\beta_1, \beta_2, \lambda)$ of functions is defined by some conditions for complex numbers $\beta_1, \beta_2 \neq 0$ and $\lambda > 0$.

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1. Introduction

Let A be the class of analytic functions on the unit disc D and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are univalent. For, $0 \leq \alpha < 1$ we denote by $\mathcal{K}(\alpha)$ the subclass of S defined by

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{S} : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \forall z \in D \right\}.$$

A function $f \in \mathcal{K}(\boldsymbol{\alpha})$ is called a convex functions of order α in D. We define the subclass $\mathcal{K}_1(\alpha)$ of $\mathcal{K}(\alpha)$ by

$$\mathcal{K}_{\mathbf{1}}(\boldsymbol{\alpha}) = \left\{ f \in \mathcal{K}(\alpha) : \frac{1}{f'(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, b_n = |b_n| e^{(in\theta)} \right\}$$

Let β_1 and $\beta_2 \neq 0$ be two complex numbers and λ be a positive real number. We denote by $\mathcal{D}(\beta_1, \beta_2, \lambda)$ the class of functions $f \in \mathcal{S}$ such that

$$|\beta_1 \left(\frac{1}{f'(z)}\right)'' + \beta_2 \left(\frac{1}{f'(z)}\right)''' | \leq \lambda, \;\; \forall z \in D \tag{1.2}$$

H. Kobashi, H. Shiraishi and S. Owa, in [4], have considered an analogous problem, for a subclass of starlike functions of order α. They have obtained some results about the radius of this subclass in the class $\mathcal{P}(\beta_1, \beta_2, \lambda)$ defined by

$$|\beta_1 \left(\frac{z}{f(z)}\right)'' + \beta_2 \left(\frac{z}{f(z)}\right)''' | \leq \lambda, \;\; \forall z \in D.$$

The present work is inspired from their work.

2. Main results

To consider our problems, we need the following technical lemma:

Lemma 2.1. If $f \in \mathcal{K}_1(\alpha)$ and

$$\frac{1}{f'(z)} = \left\{1 + \sum_{n=1}^{\infty} b_n z^n, \, b_n = |b_n| e^{in\theta}\right\}$$

then we have

$$\sum_{n=1}^{\infty} (n - 1 + \alpha)|b_n| \leq \alpha - 1$$

Proof. Let F be defined by

$$F(z) = \frac{1}{f'(z)}$$

A simple computations give us

$$\begin{aligned} Re\left(1 + \frac{zf''(z)}{f'(z)}\right) &= Re\left(1 - \frac{zF'(z)}{F(z)}\right) \\ &= Re\left(\frac{1 - \sum_{n=1}^{\infty}(n-1)b_n z^n}{1 + \sum_{n=1}^{\infty} b_n z^n}\right) \\ &= Re\left(\frac{1 - \sum_{n=1}^{\infty}(n-1)|b_n| e^{in\theta} z^n}{1 + \sum_{n=1}^{\infty} |b_n| e^{in\theta} z^n}\right) \end{aligned} \tag{2.1}$$

Since $Re\left(1 + \dfrac{zf''(z)}{f'(z)}\right) > \alpha$, then from 2.1 we obtain

$$Re\left(\frac{1 - \sum_{n=1}^{\infty}(n-1)|b_n| e^{in\theta} z^n}{1 + \sum_{n=1}^{\infty} |b_n| e^{in\theta} z^n}\right) > \alpha, \;\; \forall z \in D$$

Applying the last inequality for $z = |z|e^{-i\theta}$, we obtain that

$$\frac{1 - \sum_{n=1}^{\infty}(n-1)|b_n||z|^n}{1 + \sum_{n=1}^{\infty}|b_n||z|^n} > \alpha$$

Therefore, letting $|z| \longrightarrow 1^{-1}$, we obtain that

$$\sum_{n=1}^{\infty}(n-1+\alpha)|b_n| \leq 1-\alpha$$

■

As a consequence of 2.1, we have the following corollary

Corollary 2.2. If $f \in \mathcal{K}_1(\alpha)$, then we have

1) $|b_n| \leq \dfrac{1-\alpha}{n-1+\alpha} \leq 1, \ \forall n \geq 2$

2) $\sum_{n=1}^{\infty}(n-1)|b_n|^2 \leq 1-\alpha$

Further, we need the following technical lemma

Lemma 2.3. Let $f \in \mathcal{S}$ and $\dfrac{1}{f'(z)} = \sum_{n=1}^{\infty} b_n z^n$, $z \in D$. If f satisfies

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2|)|b_n| \leq \lambda \tag{2.2}$$

then f belongs to $\mathcal{D}(\beta_1, \beta_2, \lambda)$.

Proof. We have, for all $z \in D$

$$\begin{aligned}
&\left|\beta_1\left(\frac{1}{f'(z)}\right)'' + \beta_2\left(\frac{1}{f'(z)}\right)'''\right| \\
&\quad = \left|\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2|)b_n z^n\right| \\
&\quad \leq \sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2|)|b_n|
\end{aligned} \tag{2.3}$$

Thus if f satisfies 2.2, then f belongs to $\mathcal{D}(\beta_1, \beta_2, \lambda)$. ■

Now, we derive the main result

Theorem 2.4. Let $f \in \mathcal{K}_1(\alpha)$ and $0 < |w| < 1$. Then the function $\frac{1}{w}f(wz)$ belongs to $\mathcal{D}(\beta_1, \beta_2, \lambda)$ for $|w| < r_0$, where r_0 is the smallest positive root of the equation

$$\begin{aligned} h_\lambda(|w|) &= |\beta_1|(1-|w|)|w|^2\sqrt{2(2+|w|^2)} \\ &+ |\beta_2||w|^3\sqrt{6(3|w|^4+14|w|^2+3)} - \frac{\lambda(1-|w|^2)^3}{\sqrt{1-\alpha}} \\ &= 0 \end{aligned} \tag{2.4}$$

Proof. Let $|w| < r_0$ and let $f \in \mathcal{K}_1(\alpha)$. If we write

$$\frac{1}{f'(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n$$

, then we have

$$\frac{1}{\left(\frac{1}{w}f(wz)\right)'} = \frac{1}{f'(wz)} = 1 + \sum_{n=1}^{\infty} w^n b_n z^n \tag{2.5}$$

Thus according to the lemma 2.3, we have to prove that

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2|)|b_n||w|^n \leq \lambda \tag{2.6}$$

For the left hand of 2.6, we have

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2|)|b_n||w|^n &= |\beta_1|\sum_{n=2}^{\infty} n(n-1)|b_n||w|^n \\ &+ |\beta_2|\sum_{n=3}^{\infty} n(n-1)(n-2)|b_n||w|^n \\ &= |\beta_1|\sum_{n=2}^{\infty}(n^2(n-1)|w|^{2n})^{\frac{1}{2}}((n-1)|b_n|^2)^{\frac{1}{2}} \\ &+ |\beta_2|\sum_{n=3}^{\infty}(n^2(n-1)(n-2)^2|w|^{2n})^{\frac{1}{2}}((n-1)|b_n|^2)^{\frac{1}{2}} \end{aligned} \tag{2.7}$$

Indeed, applying Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned} &\sum_{n=2}^{\infty}(n^2(n-1)|w|^{2n})^{\frac{1}{2}}((n-1)|b_n|^2)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=2}^{\infty} n^2(n-1)|w|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}} \end{aligned} \tag{2.8}$$

and

$$\sum_{n=3}^{\infty}(n^2(n-1)(n-2)^2|w|^{2n})^{\frac{1}{2}}((n-1)|b_n|^2)^{\frac{1}{2}}$$

$$\leq \left(\sum_{n=3}^{\infty} n^2(n-1)(n-2)^2|w|^{2n}\right)^{\frac{1}{2}} \left(\sum_{n=3}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}} \tag{2.9}$$

Taking 2.8 and 2.9 in 2.7, we obtain that

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2|)|b_n||w|^n$$

$$\leq |\beta_1| \left(\sum_{n=2}^{\infty} n^2(n-1)|w|^{2n}\right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}}$$

$$+ |\beta_2| \left(\sum_{n=3}^{\infty} n^2(n-1)(n-2)^2|w|^{2n}\right)^{\frac{1}{2}} \left(\sum_{n=3}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}} \tag{2.10}$$

Applying 2) from 2.2, we obtain from 2.10 that

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2|)|b_n||w|^n$$

$$\leq |\beta_1| \left(\sum_{n=2}^{\infty} n^2(n-1)|w|^{2n}\right)^{\frac{1}{2}} (1-\alpha)^{\frac{1}{2}}$$

$$+ |\beta_2| \left(\sum_{n=3}^{\infty} n^2(n-1)(n-2)^2|w|^{2n}\right)^{\frac{1}{2}} (1-\alpha)^{\frac{1}{2}} \tag{2.11}$$

A simple computations, see for details [4], show that the two following expressions

$$\left(\sum_{n=2}^{\infty} n^2(n-1)|w|^{2n}\right)^{\frac{1}{2}} = \frac{|w|^2\sqrt{2(2+|w|^2)}}{(1-|w|^2)^2} \tag{2.12}$$

$$\left(\sum_{n=3}^{\infty} n^2(n-1)(n-2)^2|w|^{2n}\right)^{\frac{1}{2}} = \frac{|w|^3\sqrt{6(3|w|^4+14|w|^2+3)}}{(1-|w|^2)^3} \tag{2.13}$$

Taking 2.12 and 2.13 in 2.11 we obtain that

$$\begin{aligned}\sum_{n=2}^{\infty} & n(n-1)(|\beta_1| + (n-2)|\beta_2|)|b_n||w|^n \\ & \leq |\beta_1| \frac{|w|^2\sqrt{2(2+|w|^2)}}{(1-|w|^2)^2}(1-\alpha)^{\frac{1}{2}} \\ & + |\beta_2| \frac{|w|^3\sqrt{6(3|w|^4+14|w|^2+3)}}{(1-|w|^2)^3}(1-\alpha)^{\frac{1}{2}}\end{aligned} \tag{2.14}$$

Now, let $0 < |w| < 1$ satisfies

$$|\beta_1| \frac{|w|^2\sqrt{2(2+|w|^2)}}{(1-|w|^2)^2}(1-\alpha)^{\frac{1}{2}} + |\beta_2| \frac{|w|^3\sqrt{6(3|w|^4+14|w|^2+3)}}{(1-|w|^2)^3}(1-\alpha)^{\frac{1}{2}} = \lambda \tag{2.15}$$

which is equivalent to

$$\begin{aligned}h(|w|) &= |\beta_1|(1-|w|)|w|^2\sqrt{2(2+|w|^2)} \\ & + |\beta_2||w|^3\sqrt{6(3|w|^4+14|w|^2+3)} - \frac{\lambda(1-|w|^2)^3}{\sqrt{1-\alpha}} \\ &= 0\end{aligned} \tag{2.16}$$

Noting that the function h is continuous on the interval $[0, 1]$ and satisfies $h(0) = \frac{-\lambda}{\sqrt{1-\alpha}} < 0$ and $h(1) = 2|\beta_2|\sqrt{30} > 0$. This implies that the equation $h(r) = 0$ has a root in the interval $(0, 1)$. Let r_0 be the smallest positive root of $h(r) = 0$. To conclude we use the fact that the left hand of 2.15 is non-decreasing in $|w|$ and that 2.15 and 2.16 are equivalent for $|w|$ in the interval $[0, 1)$. ■

3. Remark

If λ and α satisfies

$$\lambda = \sqrt{1-\alpha}, \tag{3.1}$$

then the equation 2.16 becomes

$$|\beta_1|(1-|w|)|w|^2\sqrt{2(2+|w|^2)} + |\beta_2||w|^3\sqrt{6(3|w|^4+14|w|^2+3)} - (1-|w|^2)^3 = 0.$$

Thus in this case the value of r_0 depends only on $|\beta_1|$ and on $|\beta_2|$. In particular if λ and α satisfy 3.1 and $\beta_1 = 0$ and $|\beta_2| = 1$ we have $r_0 \approx 0.443618888$.

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