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Three Solutions for a Class of Dirichlet Quasilinear Elliptic Systems

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Abstract

In this paper, we establish the existence of at least three solutions for a class of Dirichlet quasilinear elliptic systems. The technical approach is mainly based on a recent three critical points theorem of B. Ricceri [On a three critical points theorem, Arch. Math. (Basel) 75 (2000) 220-226.].

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1. Introduction

We study the following quasilinear elliptic systems

$$\begin{cases} u_1'' + \lambda h_1(u_1')F_{u_1}(x, u_1, u_2, \ldots, u_n) = 0 & \text{in } (a, b), \\ u_2'' + \lambda h_2(u_2')F_{u_2}(x, u_1, u_2, \ldots, u_n) = 0 & \text{in } (a, b), \\ \ldots \\ u_n'' + \lambda h_n(u_n')F_{u_n}(x, u_1, u_2, \ldots, u_n) = 0 & \text{in } (a, b), \\ u_i(a) = u_i(b) = 0 \text{ for } 1 \le i \le n. \end{cases} \tag{1.1}$$

where $\lambda > 0$, $a, b \in R$, $F : [a, b] \times R^n \to R$ is a function such that $F(., t_1, \ldots, t_n)$ is continuous in $[a, b]$ for all $(t_1, \ldots, t_n) \in R^n$ and $F(x, ., \ldots, .)$ is a C^1 in R^n for almost every $x \in [a, b]$, and F_{u_i} denotes the partial derivative of F with respect to u_i.

Precisely, we deal with the existence of an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q, such that, for each $\lambda \in \Lambda$, problem (1) admits at least three solutions whose norms in $W_0^{1,2}((a,b))^n$ are less than q.

Recently, many papers have published about quasilinear elliptic systems which have been used in a great variety of application; see [1,5,8-13,18,20] and the references therein.

Since many years, the existence of multiple solutions for Dirichlet or Neumann boundary value problems has been widely investigated. Several recent results, approached by variational methods, make use of three critical points theorem of B. Ricceri ([17]) that, for example, we cite the papers [1-4,6,7,13-16].

In the present paper, our main result is Theorem 2.1 based on the following three critical points theorem proved in [17].

Theorem 1.1. Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow R$ a continuously $G\widehat{a}teaux$ differentiable and sequentially weakly lower semicontinuous functional whose $G\widehat{a}teaux$ derivative admits a continuous inverse on X^*; $\Psi : X \longrightarrow R$ a continuously $G\widehat{a}teaux$ differentiable functional whose $G\widehat{a}teaux$ derivative is compact. Assume that

$$\lim_{||u|| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$$

for all $\lambda \in [0, +\infty[$, and that there exists a continuous concave function $h : [0, \infty[\to R$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + h(\lambda)).$$

Then, there exist an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than q.

The aim of this paper is to extend the main result of [3] to the systems case.

2. Main results

Here and in the sequel, X will denote the Cartesian product of n Sobolev space $W_0^{1,2}((a,b))$, $W_0^{1,2}((a,b)), \ldots$ and $W_0^{1,2}((a,b))$, i.e., $X = W_0^{1,2}((a,b))^n$ with the norm

$$||\ (u_1, u_2, \ldots, u_n)\ || = ||u_1'||_2 + ||u_2'||_2 + \cdots + ||u_n'||_2,$$

$F : [a,b] \times R^n \to R$ is a function such that $F(., t_1, \ldots, t_n)$ is continuous in $[a,b]$ for all $(t_1, \ldots, t_n) \in R^n$ and $F(x, ., \ldots, .)$ is a C^1 in R^n for almost every $x \in [a,b]$, and F_{u_i} denotes the partial derivative of F with respect to u_i, and $h_i : R \to]0, +\infty[$ for $1 \leq i \leq n$, is a continuous function such that there exists a positive number m with $h_i(x) \geq m$ for $1 \leq i \leq n$ and for every $x \in R$.

Put

$$g_i(y) = \int_0^y (\int_0^\tau \frac{1}{h_i(\delta)} d\delta) d\tau$$

for $1 \le i \le n$ and for every $y \in R$.

Our main result is the following theorem:

Theorem 2.1. Assume that there exist $n+3$ positive constants M, c, d and γ_i for $1 \le i \le n$ with $\left(\frac{b-a}{c}\right)^2 \sum_{i=1}^{n} \left(g_i\left(\frac{4d}{b-a}\right) + g_i\left(\frac{4d}{a-b}\right)\right) > \frac{8n}{M}$, $\gamma_i < 2$ for $1 \le i \le n$ and a positive function $\mu \in L^1$ such that

(α_1) $h_i(y) \le M$ for $1 \le i \le n$ and for each $y \in R$,

(α_2) $F(x, t_1, \ldots, t_n) \ge 0$ for each $(x, t_1, \ldots, t_n) \in \left(\left[a, a + \frac{b-a}{4}\right] \cup \left[b - \frac{b-a}{4}, b\right]\right) \times [0, d]^n$,

(α_3) $\frac{M(b-a)^3}{8nc^2} \times$

$$\max_{(x,t_1,\ldots,t_n)\in[a,b]\times K} F(x, t_1, \ldots, t_n) < \frac{1}{\sum_{i=1}^{n}(g_i(\frac{4d}{b-a}) + g_i(\frac{4d}{a-b}))} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, d, \ldots, d) dx,$$

where $K = \{(t_1, \ldots, t_n) | \sum_{i=1}^{n} t_i^2 \le nc^2\}$.

(α_4) $F(x, t_1, \ldots, t_n) \le \mu(x) \left(1 + \sum_{i=1}^{n} |t_i|^{\gamma_i}\right)$ for almost every $x \in [a, b]$, for all $t_i \in R$ and for $1 \le i \le n$,

(α_5) $F(x, 0, \ldots, 0) = 0$ for almost every $x \in [a, b]$.

Then, there exist an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three solutions in $C^2([a, b])^n$ whose norms in X are less than q.

Proof. We begin by setting

$$\Phi(u) = \int_a^b \sum_{i=1}^{n} g_i(u_i'(x)) dx,$$

$$\Psi(u) = -\int_a^b F(x, u_1(x), \ldots, u_n(x)) dx$$

for each $u = (u_1, \ldots, u_n) \in X$. It is well known that Ψ is a *Gâteaux* differentiable functional whose *Gâteaux* derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)v = -\int_a^b \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx$$

for every $v = (v_1, \dots, v_n) \in X$, and that $\Psi' : X \to X^*$ is a continuous and compact operator.

Moreover, Φ is a continuously $G\widehat{a}teaux$ differentiable and sequentially weakly lower semi continuous functional whose $G\widehat{a}teaux$ derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)v = \int_a^b \sum_{i=1}^n g_i'(u_i'(x))v_i'(x)dx,$$

that Φ' admits a continuous inverse on X^*, and since classical and weak solutions to problem (1) coincide (for more details, see [14]), the solutions of problem (1) are exactly the solutions of the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0.$$

From (α_1) it follows that

$$\Phi(u_1, \dots, u_n) \geq \frac{1}{2M}(||u_1'||_2^2 + \cdots + ||u_n'||_2^2) \tag{2.1}$$

for $(u_1, \dots, u_n) \in X$ and so, thanks to (2) and (α_4), for each $\lambda > 0$ one has that

$$\lim_{||u||\to+\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$$

and so one of the assumptions of Theorem A holds. Now, we want to hold the other assumptions of Theorem A by using of Proposition 3.1 of [16], namely, we claim that there exist $r > 0$ and $w = (w_1, w_2, \dots, w_n) \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} (-\Psi(u)) < r\frac{(-\Psi(w))}{\Phi(w)}.$$

Moreover, since for $1 \leq i \leq n$,

$$\max_{x\in[a,b]} |u_i(x)| \leq \frac{(b-a)^{\frac{1}{2}}}{2}||u_i'||_2$$

for each $u_i \in W_0^{1,2}((a, b))$ (see [19]), we have $\max_{x\in[a,b]} \sum_{i=1}^n u_i^2(x) \leq \frac{b-a}{4} \sum_{i=1}^n ||u_i'||_2^2$ for each $u = (u_1, u_2, \dots, u_n) \in X$, then for each $r > 0$, (2) ensures that

$$\Phi^{-1}(] - \infty, r]) \subseteq \left\{ u = (u_1, u_2, \dots, u_n) \in X; \max \sum_{i=1}^n u_i^2(x) \leq \frac{M(b-a)r}{2} \text{ for each } x \in [a, b] \right\},$$

so

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) \leq (b-a)\max_{(x,t_1,\dots,t_n)\in[a,b]\times K} F(x,t_1,\dots,t_n).$$

Put $w(x) = (w_1(x), w_2(x), \dots, w_n(x))$ such that for $1 \leq i \leq n$,

$$w_i(x) = \begin{cases} \dfrac{4}{b-a}d(x-a) & \text{if } a \leq x < a + \dfrac{b-a}{4}, \\ d & \text{if } a + \dfrac{b-a}{4} \leq x \leq b - \dfrac{b-a}{4}, \\ \dfrac{4}{b-a}d(b-x) & \text{if } b - \dfrac{b-a}{4} < x \leq b. \end{cases}$$

It is clear to see that $w \in X$ and $\Phi(w) = \dfrac{b-a}{4}\displaystyle\sum_{i=1}^{n}\left(g_i\left(\frac{4d}{b-a}\right) + g_i\left(\frac{4d}{a-b}\right)\right)$.

Moreover, thanks to assumptions (α_2) and (α_3), and by choosing $r = \dfrac{2nc^2}{M(b-a)}$, one has $\Phi(w) > r$ and

$$\begin{aligned}(b-a)&\max_{(x,t_1,\dots,t_n)\in[a,b]\times K} F(x,t_1,\dots,t_n)\\ &< \frac{\frac{2nc^2}{M(b-a)}}{\frac{b-a}{4}\sum_{i=1}^{n}(g_i(\frac{4d}{b-a}) + g_i(\frac{4d}{a-b}))}\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d,\dots,d)dx\\ &\leq r\frac{(-\Psi(w))}{\Phi(w)}.\end{aligned}$$

Namely

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < r\frac{(-\Psi(w))}{\Phi(w)}.$$

Fix ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < \rho < r\frac{(-\Psi(w))}{\Phi(w)}$$

and define $h(\lambda) = \lambda\rho$ for every $\lambda \geq 0$, from Proposition 3.1 of [16], with $x_0 = 0,\ x_1 = w,\ J = -\Psi$ we obtain

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Now, our conclusion follows from Theorem 1.1. ■

Let f be a continuous function in $[a,b]$ and $\widetilde{f_i}$ for $1 \leq i \leq n$, be a functions in C^1 and

$$F(x,u_1,\dots,u_n) = f(x)\left(\prod_{i=1}^{n}\widetilde{f_i}(u_i)\right)$$

for each $(x, u_1, \dots, u_n) \in [a, b] \times R^n$. Then, by using the Theorem 2.1, we have the following result:

Corollary 2.2. Assume that there exist $n+3$ positive constants M, c, d and γ_i for $1 \le i \le n$ with $\left(\frac{b-a}{c}\right)^2 \sum_{i=1}^{n} \left(g_i\left(\frac{4d}{b-a}\right) + g_i\left(\frac{4d}{a-b}\right)\right) > \frac{8n}{M}$, $\gamma_i < 2$ for $1 \le i \le n$ and a positive function $\mu \in L^1$ such that

(α'_1) $h_i(y) \le M$ for $1 \le i \le n$ and for each $y \in R$,

(α'_2) $f(x) \ge 0$ for each $x \in \left[a, a + \frac{b-a}{4}\right] \cup \left[b - \frac{b-a}{4}, b\right]$ and $\widetilde{f}_i(t_i) \ge 0$ for each $t_i \in [0, d]$, $1 \le i \le n$,

(α'_3) $\frac{M(b-a)^3}{8nc^2} \times$

$$\max_{(x,t_1,\dots,t_n) \in [a,b] \times K} \left(f(x)\left(\prod_{i=1}^{n} \widetilde{f}_i(t_i)\right)\right) < \frac{\prod_{i=1}^{n} \widetilde{f}_i(d)}{\sum_{i=1}^{n}\left(g_i\left(\frac{4d}{b-a}\right) + g_i\left(\frac{4d}{a-b}\right)\right)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} f(x)dx,$$

where $K = \left\{(t_1, \dots, t_n) \mid \sum_{i=1}^{n} t_i^2 \le nc^2\right\}$.

(α'_4) $f(x)\left(\prod_{i=1}^{n} \widetilde{f}_i(t_i)\right) \le \mu(x)\left(1 + \sum_{i=1}^{n} |t_i|^{\gamma_i}\right)$ for almost every $x \in [a, b]$ and for all $t_i \in R$, $1 \le i \le n$,

(α'_5) $\widetilde{f}_i(0) = 0$ for $1 \le i \le n$.

Then, there exist an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem

$$\begin{cases} u_1'' + \lambda h_1(u_1') f(x) \widetilde{f}_1'(u_1)\left(\prod_{i=1, i\neq 1}^{n} \widetilde{f}_i(u_i)\right) = 0 & \text{in } (a, b), \\ u_2'' + \lambda h_2(u_2') f(x) \widetilde{f}_2'(u_2)\left(\prod_{i=1, i\neq 2}^{n} \widetilde{f}_i(u_i)\right) = 0 & \text{in } (a, b), \\ \dots \\ u_n'' + \lambda h_n(u_n') f(x) \widetilde{f}_n'(u_n)\left(\prod_{i=1, i\neq n}^{n} \widetilde{f}_i(u_i)\right) = 0 & \text{in } (a, b), \\ u_i(a) = u_i(b) = 0 \text{ for } 1 \le i \le n. \end{cases} \tag{2.2}$$

admits at least three solutions in $C^2([a, b])^n$ whose norms in X are less than q.

Now, we want to point out a remarkable consequence of Theorem 2.1:

Theorem 2.3. Let $F : R^n \to R$ be a C^1 function and assume that there exist $n+4$ positive constants M, c, d, η and γ_i for $1 \le i \le n$ with

$$\left(\frac{b-a}{c}\right)^2 \sum_{i=1}^{n} \left(g_i\left(\frac{4d}{b-a}\right) + g_i\left(\frac{4d}{a-b}\right)\right) > \frac{8n}{M}$$

and $\gamma_i < 2$ for $1 \le i \le n$ such that
(β_1) $h_i(y) \le M$ for $1 \le i \le n$ and for each $y \in R$,
(β_2) $F(t_1, \dots, t_n) \ge 0$ for each $(t_1, \dots, t_n) \in [0, d]^n$,
(β_3) $\dfrac{M(b-a)^2}{4nc^2} \max_{(t_1,\dots,t_n)\in K} F(t_1, \dots, t_n) < \dfrac{F(d, \dots, d)}{\sum_{i=1}^{n}(g_i(\frac{4d}{b-a}) + g_i(\frac{4d}{a-b}))}$,
where $K = \left\{(t_1, \dots, t_n) | \sum_{i=1}^{n} t_i^2 \le nc^2\right\}$.
(β_4) $F(t_1, \dots, t_n) \le \eta \left(1 + \sum_{i=1}^{n} |t_i|^{\gamma_i}\right)$ for all $t_i \in R$, $1 \le i \le n$,
(β_5) $F(0, \dots, 0) = 0$.

Then, there exist an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem

$$\begin{cases} u_1'' + \lambda h_1(u_1') F_{u_1}(u_1, u_2, \dots, u_n) = 0 & \text{in } (a, b), \\ u_2'' + \lambda h_2(u_2') F_{u_2}(u_1, u_2, \dots, u_n) = 0 & \text{in } (a, b), \\ \cdots & \\ u_n'' + \lambda h_n(u_n') F_{u_n}(u_1, u_2, \dots, u_n) = 0 & \text{in } (a, b), \\ u_i(a) = u_i(b) = 0 \text{ for } 1 \le i \le n. & \end{cases} \tag{2.3}$$

admits at least three solutions in $C^2([a, b])^n$ whose norms in X are less than q.

Remark 2.4. If $F : [a, b] \times R \to R$, namely $n = 1$, the results give back the same results obtained in [3].

Let us present now an example of satisfying the assumptions of Theorem 2.3:

Example 2.5. Consider the problem

$$\begin{cases} u_1'' + \dfrac{\lambda}{1+|u_1'|} e^{-u_1} u_1^8 (9 - u_1) = 0 & \text{in } (-1, 3), \\ u_2'' + \dfrac{\lambda}{1+2|u_2'|} e^{-u_2} u_2^8 (9 - u_2) = 0 & \text{in } (-1, 3), \\ u_3'' + \dfrac{\lambda}{1+3|u_3'|} e^{-u_3} u_3^8 (9 - u_3) = 0 & \text{in } (-1, 3), \\ u_i(-1) = u_i(3) = 0 \ \text{ for } 1 \le i \le 3. & \end{cases} \tag{2.4}$$

With $F(u_1, u_2, u_3) = e^{-u_1}u_1^9 + e^{-u_2}u_2^9 + e^{-u_3}u_3^9$ and $h(u_i') = \dfrac{1}{1+i|u_i'|}$ for each $u_i \in R$ that $1 \le i \le 3$, all the assumptions of Theorem 2.3 since

$$\max_{u_1^2+u_2^2+u_3^2 \le \frac{3}{4}} (e^{-u_1}u_1^9 + e^{-u_2}u_2^9 + e^{-u_3}u_3^9) \le 3 \max_{u_i^2 \le \frac{3}{4}} (e^{-u_i}u_i^9)$$

for $1 \le i \le 3$, are satisfied by choosing, for instance $M = 1$, $c = \dfrac{1}{2}$, $d = 2$, $\gamma = 1$ and η sufficiently large. So there exist an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (5) admits at least three solutions in $C^2([a, b])^3$ whose norms in $W_0^{1,2}([-1, 3])^3$ are less than q.

Finally, we conclude this paper with the following very particular case of Theorem 2.3.

Let $h_i(y) = 1$ for $1 \le i \le n$ and for every $y \in R$. So we have:

Theorem 2.6. Let $F : R^n \to R$ be a C^1 function and assume that there exist $n + 3$ positive constants c, d, η and γ_i for $1 \le i \le n$ with $2d > c$ and $\gamma_i < 2$ for $1 \le i \le n$ such that

(δ_1) $F(t_1, \dots, t_n) \ge 0$ for each $(t_1, \dots, t_n) \in [0, d]^n$,

(δ_2) $\displaystyle\max_{(t_1,\dots,t_n)\in K} F(t_1, \dots, t_n) < \frac{1}{8}\left(\frac{c}{d}\right)^2 F(d, \dots, d)$,

where $K = \left\{(t_1, \dots, t_n) | \displaystyle\sum_{i=1}^n t_i^2 \le nc^2\right\}$.

(δ_3) $F(t_1, \dots, t_n) \le \eta \left(1 + \displaystyle\sum_{i=1}^n |t_i|^{\gamma_i}\right)$ for all $t_i \in R$, $1 \le i \le n$,

(δ_4) $F(0, \dots, 0) = 0$.

Then, there exist an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem

$$\begin{cases} u_1'' + \lambda F_{u_1}(u_1, u_2, \dots, u_n) = 0 & \text{in } (a, b), \\ u_2'' + \lambda F_{u_2}(u_1, u_2, \dots, u_n) = 0 & \text{in } (a, b), \\ \dots & \\ u_n'' + \lambda F_{u_n}(u_1, u_2, \dots, u_n) = 0 & \text{in } (a, b), \\ u_i(a) = u_i(b) = 0 \text{ for } 1 \le i \le n. & \end{cases} \tag{2.5}$$

admits at least three solutions in $C^2([a, b])^n$ whose norms in X are less than q.

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