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Cohomology of unbranched Riemann domains over q-complete spaces

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Abstract

In this article, we show that if $\Pi : X \to Y$ is an unbranched Riemann domain over an r-complete complex space Y, then for any coherent analytic sheaf $\mathcal{F}$ on X the cohomology group $H^p(X, \mathcal{F})$ vanishes for all $p \geq q+r-1$. In particular, one gets a positive answer to a generalization of the local Stein problem and new vanishing theorems for the cohomology of locally q-complete open sets in r-complete spaces.

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1. Introduction

A holomorphic map $\Pi : X \to Y$ of complex spaces is said to be a locally q-complete morphism if for every $x \in Y$, there exists an open neighborhood U of x such that $\Pi^{-1}(U)$ is q-complete. When $q = 1$, then Π is called a locally 1-complete or locally Stein morphism.

The map Π is said to be r-complete if there exists a smooth function $\phi : X \to \mathbb{R}$ such that:

(a) ϕ is r-convex on X

(b) For every $\lambda \in \mathbb{R}$, the restriction map $\Pi : \{x \in X : \phi(x) \leq \lambda\} \to Y$ is proper.

In [3], Andreotti and Narasimhan proved that if Y is a Stein space with isolated singularities and $X \subset Y$ is locally Stein in Y, then X is Stein.

In [5], Coltoiu and Diederich have shown that if $\Pi : X \to Y$ is an unbranched Riemann domain with Y a Stein space and Π a locally Stein morphism, then X is Stein if Y has isolated singularities. This result clearly generalizes the theorem of Andreotti and Narasimhan proved when $X \subset Y$ and Π is the inclusion map.

In [1], we have proved that the above mentioned result of Coltoiu and Diederich follows if we assume only that Y is an arbitrary Stein space.

It is also known from [8] that if X and Y are complex spaces such that there exists a locally q-complete morphism $\Pi : X \to Y$, then Y r-complete implies X cohomologically $(q+r)$-complete. The aim of this article is to prove that if moreover $\Pi : X \to Y$ is an unbranched Riemann domain, then X is cohomologically $(q+r-1)$-complete.

This result has been shown in [1] in the special case $q = r = 1$ and extends in particular a theorem in [2] which is proved in the regular case for $r = 1$ and $\mathcal{F} = \mathcal{O}_X$, namely; if $X \subset Y$ is locally q-complete in a Stein manifold Y of dimension n and $1 \leq q \leq n-2$, then X is cohomologically q-complete with respect to the structure sheaf $\mathcal{O}_X$.

As an immediate consequence of our result is that if X is a locally q-complete open subset of a r-complete complex space Y, then $H^p(X, \mathcal{F})$ vanishes for every coherent analytic sheaf $\mathcal{F}$ on X and all $p \geq q+r-1$.

2. Preliminaries

Recall that a smooth real valued function ϕ on a complex space X is called q-convex if every point $x \in X$ has an open neighborhood U isomorphic to a closed analytic set in a domain $D \subset \mathbb{C}^n$ such that the restriction $\phi|_U$ has an extension $\tilde{\phi} \in C^\infty(D)$ whose Levi form

$$L_z(\phi, \xi) = \sum_{i,j} \frac{\partial^2 \tilde{\phi}(z)}{\partial z_i \partial \overline{z}_j} \xi_i \overline{\xi}_j,\ \xi \in \mathbb{C}^n,$$

has at most $(q-1)$-negative or zero eigenvalues at every point $z \in D$.

The complex space X is called q-complete if X has an exhaustion function $\phi \in C^\infty(X)$ which is q-convex on X.

We say that X is cohomologically q-complete if for every coherent analytic sheaf $\mathcal{F}$ on X the cohomology groups $H^p(X, F)$ vanish for all $p \geq q$.

It is well known from the theory of Andreotti and Grauert [3] that every q-complete space is cohomologically q-complete.

An open subset D of X is called q-Runge if for every compact set $K \subset D$, there is a q-convex exhaustion function $\phi \in C^\infty(\Omega)$ such that

$$K \subset \{x \in \Omega : \phi(x) < 0\} \subset\subset D$$

In [3], Andreotti-Grauert proved that if D is q-Runge in X, then for every $\mathcal{F} \in coh(X)$

the cohomology groups $H^p(D, \mathcal{F})$ vanish for $p \geq q$ and, the restriction map

$$H^p(X, \mathcal{F}) \longrightarrow H^p(D, \mathcal{F})$$

has dense image for all $p \geq q - 1$.

3. Main result

Lemma 3.1. Let X and Y be complex spaces and $\Pi : X \to Y$ an unbranched Riemann domain. Assume that there exists a smooth r-convex function ϕ on Y. Then, for any real number c and every coherent analytic sheaf $\mathcal{F}$ on X if $prof(\mathcal{F}) > r$ and $X'_c = \{x \in X : \phi o \Pi(x) > c\}$, the restriction map $H^p(X, \mathcal{F}) \to H^p(X'_c, \mathcal{F})$ is an isomorphism for $p \leq prof(\mathcal{F}) - r - 1$.

For the proof of Lemma 3.1 see [1].

Let now $\Pi : X \to Y$ be a locally q-complete morphism with Y r-complete, and let $\phi : Y \to \mathbb{R}$ be a smooth exhaustion function which is r-convex on Y. We fix a real number $\lambda_0 \in \mathbb{R}$. Then there exist a Stein open covering $U_1, \cdots, U_k \subset\subset Y$ of $\{\phi = \lambda_0\}$ such that $\Pi^{-1}(U_i)$ are q-complete. We choose compactly supported functions $\theta_j \in C^\infty(U_j, \mathbb{R}^+)$ such that $\sum_{j=1}^{k} \theta_j(x) > 0$ at any point $x \in \{\phi = \lambda_0\}$. There exist sufficiently small constants $c_1 > 0, \cdots, c_k > 0$ such that the functions $\phi_j = \phi - \sum_{i=1}^{j} c_i \theta_i$ are still r-convex for $1 \leq j \leq k$. We define for every integer $j = 0, \cdots, k$, the sets

$$Y_j = \{z \in Y : \phi_j(z) < \lambda_0\}, \text{ and } X_j = \Pi^{-1}(Y_j),$$

where $\phi_0 = \phi$ and, for every real number $\lambda \in \mathbb{R}$ we put

$$Y(\lambda) = \{y \in Y : \phi(y) < \lambda\} \text{ and } X(\lambda) = \Pi^{-1}(Y(\lambda)).$$

Let now $\mathcal{F}$ be a coherent analytic sheaf on X and A the set of all real numbers μ such that $H^i(X(\lambda), \mathcal{F}) = 0$ for every real number $\lambda \leq \mu$ and all $i \geq q + r - 1$.

Lemma 3.2. The set $A \subset \mathbb{R}$ is not empty and open.

Proof. First we prove that the lemma holds when $r \geq 3$. Since ϕ is proper, then obviously A is not empty. To prove that A is open, it is sufficient to show that if $\lambda_0 \in A$, then there exists $\varepsilon_0 > 0$ such that $\lambda_0 + \varepsilon_0 \in A$. In fact, since $Y_0 = Y(\lambda_0) \subset\subset Y_k$ and ϕ is exhaustive on Y, there exists $\varepsilon_0 > 0$ such that $Y(\lambda_0 + \varepsilon_0) \subset Y_k$. Next, we put $p = q + r - 1$ and consider the sets $X_j(\lambda)$ defined for every real number λ with $\lambda_0 \leq \lambda \leq \lambda_0 + \varepsilon_0$ by

$$X_j(\lambda) = X(\lambda) \cap X_j \text{ for } j = 0, \cdots, k$$

We first show by induction on j that

$$H^i(X_j(\lambda),\mathcal{F})=0 \text{ for all } i\geq p \text{ and } j=0,\cdots,k$$

This is clearly satisfied for $j=0$, since $X_0(\lambda)=X(\lambda_0)$ and $\lambda_0\in A$. Suppose now that $1\leq j\leq k$ and that $H^i(X_{j-1}(\lambda),\mathcal{F})=0$ for all $i\geq p$. Since $Y_j=Y_{j-1}\cup(Y_j\cap U_j)$, then $X_j(\lambda)=X_{j-1}(\lambda)\cup V_j(\lambda)$, where $V_j(\lambda)=X_j(\lambda)\cap\Pi^{-1}(U_j)$. Note also that

$$X_{j-1}(\lambda)\cap V_j(\lambda)=\{x\in\Pi^{-1}(U_j):\phi o\Pi(x)<\lambda,\phi_{j-1}o\Pi(x)<\lambda_0\}$$

is clearly p-Runge in $\Pi^{-1}(U_j)$. (See for instance [8]). Then the image of $H^{p-1}(\Pi^{-1}(U_j),\mathcal{F})$ is dense in $H^{p-1}(X_{j-1}(\lambda)\cap V_j(\lambda),\mathcal{F})$. Moreover, the restriction map

$$\Pi|_{X_{j-1}(\lambda)\cap V_j(\lambda)}:X_{j-1}(\lambda)\cap V_j(\lambda)\to Y_{j-1}\cap Y(\lambda)\cap U_j$$

is clearly a q-complete morphism. In fact, if ψ is a smooth q-convex exhaustion function on $\Pi^{-1}(U_j)$, then for every $c\in\mathbb{R}$, the restriction of $\Pi|_{X_{j-1}(\lambda)\cap V_j(\lambda)}$ from $\{x\in X_{j-1}(\lambda)\cap V_j(\lambda):\psi(x)\leq c\}$ to $Y_{j-1}\cap Y(\lambda)\cap U_j$ is proper. It then follows from ([6], p. 995) that $H^{p-1}(X_{j-1}(\lambda)\cap V_j(\lambda),\mathcal{F})$ is separated, which implies that $H^i(X_{j-1}(\lambda)\cap V_j(\lambda),\mathcal{F})=0$ for all $i\geq p-1$, since $\Pi^{-1}(U_j)$ is q-complete, $H^i(\Pi^{-1}(U_j),\mathcal{F})\to H^i(X_{j-1}(\lambda)\cap V_j(\lambda),\mathcal{F})$ has a dense image for $i\geq p-1$ and $p-1\geq q+1$.

We now consider the Mayer-Vietoris sequence for cohomology

$$\begin{aligned}\cdots&\to H^{p-1}(X_{j-1}(\lambda)\cap V_j(\lambda),\mathcal{F})\to H^p(X_j(\lambda),\mathcal{F})\\&\to H^p(X_{j-1}(\lambda),\mathcal{F})\oplus H^p(V_j(\lambda),\mathcal{F})\\&\to H^p(X_{j-1}(\lambda)\cap V_j(\lambda),\mathcal{F})\to\cdots\end{aligned}$$

Since $V_j(\lambda)=\{x\in\Pi^{-1}(U_j):\phi o\Pi(x)<\lambda,\phi_j o\Pi(x)<\lambda_0\}$ is clearly p-complete and $H^i(X_{j-1}(\lambda)\cap V_j(\lambda),\mathcal{F})=0$ for $i\geq p-1$, then

$$H^p(X_j(\lambda),\mathcal{F})\cong H^p(X_{j-1}(\lambda),\mathcal{F})=0$$

If we take $j=k$, we find that $X_k(\lambda)=X(\lambda)$. Therefore $H^p(X(\lambda),\mathcal{F})=0$ for all $\lambda\leq\lambda_0+\varepsilon_0$ and $\lambda_0+\varepsilon_0\in A$.

Now suppose that $r=2$. Then in view of the previous demonstration we have to prove only that if $\lambda_0\in A$ and $\varepsilon_0>0$ with $X(\lambda_0+\varepsilon_0)\subset X_k$, then $H^{q+1}(X_j(\lambda),\mathcal{F})=0$ for every $\lambda\in\mathbb{R}$ with $\lambda_0\leq\lambda\leq\lambda_0+\varepsilon_0$ and all integer $j=0,\cdots,k$. Since for every $\lambda\leq\lambda_0$, $X_0(\lambda)=X(\lambda)$ and $H^i(X_0(\lambda),\mathcal{F})=0$ for $i\geq p$, so we assume that $1\leq j\leq k$ and $H^i(X_{j-1}(\lambda),\mathcal{F})=0$ for all $i\geq p$ and $\lambda_0\leq\lambda\leq\lambda_0+\varepsilon_0$.

Now consider the long exact sequence of cohomology

$$\begin{aligned}\cdots\to H^q(X_{j-1}(\lambda),\mathcal{F})\oplus H^q(V_j(\lambda),\mathcal{F})&\xrightarrow{r^*}H^q(X_{j-1}(\lambda)\cap V_j(\lambda),\mathcal{F})\\\to H^{q+1}(X_j(\lambda),\mathcal{F})&\to H^{q+1}(X_{j-1}(\lambda),\mathcal{F})\oplus H^{q+1}(V_j(\lambda),\mathcal{F})\end{aligned}$$

Since $V_j(\lambda)$ is p-complete and $X_{j-1}(\lambda) \cap V_j(\lambda)$ is p-Runge in $\Pi^{-1}(U_j)$, then $H^{q+1}(V_j(\lambda), \mathcal{F}) = 0$ and $H^q(V_j(\lambda), \mathcal{F}) \to H^q(X_{j-1}(\lambda) \cap V_j(\lambda), \mathcal{F})$ has dense image. Moreover, since $H^{q+1}(X_{j-1}(\lambda), \mathcal{F}) = 0$ by the inductive hypothesis, then one gets the exact sequence

$$\cdots \to H^q(X_{j-1}(\lambda), \mathcal{F}) \oplus H^q(V_j(\lambda), \mathcal{F}) \xrightarrow{r^*} H^q(X_{j-1}(\lambda) \cap V_j(\lambda), \mathcal{F}) \\ \to H^{q+1}(X_j(\lambda), \mathcal{F}) \to 0$$

Because the map r^* has dense image, we conclude that $H^{q+1}(X_j(\lambda), \mathcal{F}) = 0$. ■

Lemma 3.3. Let $\Pi : X \to Y$ be an unbranched Riemann domain over an r-complete space Y. Suppose that Π is a locally q-complete morphism. Then for any coherent analytic sheaf $\mathcal{F}$ on X with $prof(\mathcal{F}) \geq 2r + q$, the cohomology groups $H^p(X, \mathcal{F})$ vanish for all $p \geq q + r - 1$.

Proof. Let $A \subset \mathbb{R}$ be the set of all real numbers μ such that $H^i(X(\lambda), \mathcal{F}) = 0$ for every real number $\lambda \leq \mu$ and all $i \geq q + r - 1$. Then in order to prove Lemma 3.3 we have only to verify that

(a) A is closed

(b) for every pair of real numbers $\lambda < \beta$, the restriction map

$$H^{p-1}(X(\beta), \mathcal{F}) \to H^{p-1}(X(\alpha), \mathcal{F})$$

has dense image.

For this, suppose first that $r \geq 3$ and fix some $\lambda_0 \in A$. Then $H^{p-1}(X(\lambda), \mathcal{F}) = 0$ for $\lambda \leq \lambda_0$. In fact, let $\varepsilon_0 > 0$ be such that $\lambda_0 + \varepsilon_0 \in A$, and define for every real number λ with $\lambda_0 \leq \lambda \leq \lambda_0 + \varepsilon_0$ and integer $j = 0, \cdots, k$, the open sets $X'_j(\lambda) = \{x \in \Pi^{-1}(U_j) : \phi o \Pi(x) > \lambda\}$. Then, by lemma 1,

$$H^l(X'_j(\lambda), \mathcal{F}) \cong H^l(\Pi^{-1}(U_j), \mathcal{F}) \text{ for } l \leq prof(\mathcal{F}) - r - 1.$$

Furthermore, since $X'_j(\lambda) = \{x \in \Pi^{-1}(U_j) \cup X_j(\lambda) : \phi o \Pi(x) > \lambda\}$, then, if $S_j = \{x \in \Pi^{-1}(U_j) \cup X_j(\lambda) : \phi o \Pi(x) \leq \lambda\}$, the cohomology group $H^l_{S_j}(\Pi^{-1}(U_j) \cup X_j(\lambda), \mathcal{F})$ vanishes for $l \leq prof(\mathcal{F}) - r$. (See [1], proof of Lemma 3.3). Moreover, the exact sequence of local cohomology

$$\cdots \to H^l_{S_j}(\Pi^{-1}(U_j) \cup X_j(\lambda), \mathcal{F}) \\ \to H^l(\Pi^{-1}(U_j) \cup X_j(\lambda), \mathcal{F}) \to H^l(X'_j(\lambda), \mathcal{F}) \\ \to H^{l+1}_{S_j}(\Pi^{-1}(U_j) \cup X_j(\lambda), \mathcal{F}) \to \cdots$$

implies that $H^l(\Pi^{-1}(U_j)\cup X_j(\lambda),\mathcal{F})\cong H^l(X'_j(\lambda),\mathcal{F})$ for all l with $l\leq prof(\mathcal{F})-r-1$ and, therefore $H^l(\Pi^{-1}(U_j)\cup X_j(\lambda),\mathcal{F})\cong H^l(\Pi^{-1}(U_j),\mathcal{F})$ for $l\leq prof(\mathcal{F})-r-1$.

Now by using the Mayer-Vietoris sequence for cohomology

$$\cdots\to H^l(\Pi^{-1}(U_j)\cup X_j(\lambda),\mathcal{F})\to H^l(X_j(\lambda),\mathcal{F})\oplus H^l(\Pi^{-1}(U_j),\mathcal{F})$$
$$\to H^l(\Pi^{-1}(U_j)\cap X_j(\lambda),\mathcal{F})\to H^{l+1}(\Pi^{-1}(U_j)\cup X_j(\lambda),\mathcal{F})\to\cdots$$

and the fact that $\Pi^{-1}(U_j)$ is q-complete and $H^l(\Pi^{-1}(U_j)\cup X_j(\lambda),\mathcal{F})\cong H^l(\Pi^{-1}(U_j),\mathcal{F})$ for $l\leq prof(\mathcal{F})-r-1$, we find that $H^l(X_j(\lambda),\mathcal{F})\cong H^l(\Pi^{-1}(U_j)\cap X_j(\lambda),\mathcal{F})$ for all l with $q\leq l\leq prof(\mathcal{F})-r-2$. Since $q\leq p-1\leq prof(\mathcal{F})-r-2$ and $H^{p-1}(\Pi^{-1}(U_j)\cap X_j(\lambda),\mathcal{F})=0$, $\Pi^{-1}(U_j)\cup X_j(\lambda)$ being p-Runge in $\Pi^{-1}(U_j)$, the restriction map

$$\Pi|_{X_j(\lambda)\cap\Pi^{-1}(U_j)}:X_j(\lambda)\cap\Pi^{-1}(U_j)\to Y_j(\lambda)\cap U_j$$

is q-complete and $p-1\geq q+1$, then

$$H^{p-1}(X(\lambda),\mathcal{F})=H^{p-1}(X_k(\lambda),\mathcal{F})=0.$$

Therefore Lemma 3.3 follows from Lemma 3.2 and the approximation result in ([3], p. 250).

Now suppose that $r=2$. We shall first prove assertion (b). For this, we consider the set T of all real numbers β such that the restriction map

$$H^q(X(\beta),\mathcal{F})\to H^q(X(\lambda),\mathcal{F})$$

has dense image for every real number λ with $\lambda<\beta$. Then clearly T is not empty. Also T is closed according to an approximation lemma on Fréchet spaces due to Andreotti and Grauert (See [3], p. 246). To prove that T is open, it is sufficient to show that if $\lambda_0\in T$ and $\varepsilon_0>0$ with $X(\lambda_0+\varepsilon_0)\subset X_k$, then for any pair of real numbers (λ,β) with $\lambda_0\leq\lambda<\beta\leq\lambda_0+\varepsilon_0$, the restriction map

$$H^q(X_j(\beta),\mathcal{F})\to H^q(X_j(\lambda),\mathcal{F})$$

has dense image for $j=0,\cdots,k$.

For this, we consider the long exact sequence of cohomology

$$\cdots\to H^q(\Pi^{-1}(U_j)\cup X_j(\lambda),\mathcal{F})\to H^q(X_j(\lambda),\mathcal{F})\oplus H^q(\Pi^{-1}(U_j),\mathcal{F})$$
$$\to H^q(V_j(\lambda),\mathcal{F})\to H^{q+1}(\Pi^{-1}(U_j)\cup X_j(\lambda),\mathcal{F})\to\cdots$$

Since we have shown that $H^l(\Pi^{-1}(U_j)\cup X_j(\lambda),\mathcal{F})\cong H^l(\Pi^{-1}(U_j),\mathcal{F})=0$ for $q\leq l\leq prof(\mathcal{F})-3$, we deduce from the long exact sequence of cohomology that $H^q(X_j(\lambda),\mathcal{F})\cong H^q(V_j(\lambda),\mathcal{F})$. If we fix $\lambda,\beta\in\mathbb{R}$ with $\lambda_0\leq\lambda<\beta\leq\lambda_0+\varepsilon_0$, since $V_j(\lambda)$ is p-Runge in $\Pi^{-1}(U_j)$ and $V_j(\lambda)\subset V_j(\beta)\subset\Pi^{-1}(U_j)$, then it follows that $H^q(X_j(\beta),\mathcal{F})\to H^q(X_j(\lambda),\mathcal{F})$ has dense range. This proves statement (b).

It is now easy to see that assertion (a) follows as an immediate consequence of Proposition 20 of [3] and Lemma 3.2 of [8]. ■

Lemma 3.4. Let X be a complex space, Y a complex manifold of dimension n and let $\Pi : X \times Y \to X$ be the first projection. Then for any coherent analytic sheaf $\mathcal{F}$ on X it follows that $prof(\Pi^*(\mathcal{F})) \geq n$.

Proof. In fact, let $n_x = emb_x(X)$ denote the embedding dimension of X at an arbitrary point $x \in X$. Then there exists a neighborhood U of x and a biholomorphic map $f : U \to U_1$ onto a closed complex subspace U_1 of a domain D in $\mathbb{C}^{n_x}$. Let $\hat{\mathcal{F}}$ be the trivial extension of $f_*(\mathcal{F}|_U)$ to D. Then we have on a neighborhood of $f(x)$ a resolution of $\hat{\mathcal{F}}$ of minimal length

$$0 \to \mathcal{O}_D^{p_d} \to \cdots \to \mathcal{O}_D^{p_0} \to \hat{\mathcal{F}} \to 0$$

Let $y \in Y$, and choose a neighborhood V of y that can be identified to a domain in $\mathbb{C}^n$. Then the map $h : U \times V \to U_1 \times V$ defined by $h(x, y) = (f(x), y)$ is biholomorphic onto the closed subspace $U_1 \times V$ of $\Omega = D \times V \subset \mathbb{C}^{n(x)} \times \mathbb{C}^n$. If $\mathcal{G}$ is the trivial extension of $h_*(\Pi^*(\mathcal{F})|_{U\times V})$ to Ω and $\tilde{\Pi} : \Omega = D \times V \to D$ the natural projection, then $\mathcal{G} \cong \tilde{\Pi}^*(\hat{\mathcal{F}})$. Therefore the sequence

$$0 \to \mathcal{O}_D^{p_d} \to \cdots \to \mathcal{O}_D^{p_0} \to \hat{\mathcal{F}} \to 0$$

is transformed by $\tilde{\Pi}^*$ into the exact Ω-sequence

$$\mathcal{O}_\Omega^{p_d} \to \cdots \to \mathcal{O}_\Omega^{p_0} \to \mathcal{G} \to 0$$

Since the minimal number of generators of $\mathfrak{m}_x = \mathfrak{m}(\mathcal{O}_{D,x})$ is equal to $n_x = emb_x(X)$, then $\mathcal{O}_D^{p_l} = 0$ for all $l > n_x$ by a theorem of Hilbert. (cf [3], p. 194). Therefore $\mathcal{O}_\Omega^{p_l} = \tilde{\Pi}^*(\mathcal{O}_D^{p_l}) = 0$ if $l > n_x$, which implies that $prof_{(x,y)}(\Pi^*(\mathcal{F})) \geq n$. ■

Theorem 3.5. Let X and Y be complex spaces and $\Pi : X \to Y$ an unbranched Riemann domain. Assume that Y is r-complete and Π a locally q-complete morphism. Then for every $\mathcal{F} \in Coh(X)$ it follows that $H^p(X, \mathcal{F}) = 0$ for $p \geq q + r - 1$.

Proof. If $prof(\mathcal{F}) \geq 2r + q$, then the theorem follows from lemma 3. We identify X with $X \times \{z_0\}$ for some $z_0 \in \mathbb{C}^n$ with $n \geq 2r + q$ and denote by $\Pi_1 : X \times \mathbb{C}^n \to X$ the first projection.

Clearly the map $X \times \mathbb{C}^n \xrightarrow{\tilde{\Pi}} Y \times \mathbb{C}^n$ defined by $\tilde{\Pi}(x, y) = (\Pi(x), y)$ is an unbranched Riemann domain and locally q-complete. Since $Y \times \mathbb{C}^n$ is r-complete and $prof(\Pi_1^*\mathcal{F}) \geq 2r + q$ by lemma 3, it follows from lemma 2 that $H^p(X \times \mathbb{C}^n, \Pi_1^*(\mathcal{F})) = 0$ for all $p \geq q + r - 1$.

Let $\mathcal{I}(X)$ be the sheaf of germs of holomorphic functions which vanish on X, $\mathcal{O}_X = \mathcal{O}_{X\times\mathbb{C}^n}/\mathcal{I}(X)$ and $\Pi_1^*(\mathcal{F})_X = \Pi_1^*(\mathcal{F}) \otimes_{\mathcal{O}_{X\times\mathbb{C}^n}} \mathcal{O}_X$. If e is the image in $\mathcal{O}_X$ of the section 1 of $\mathcal{O}_{X\times\mathbb{C}^n}$, then any element of $\Pi_1^*(\mathcal{F})_{X,x}$ can be written in the form $\alpha \otimes e_x$

where $\alpha \in \Pi_1^*(\mathcal{F})_x$. Let $\eta : \Pi_1^*(\mathcal{F}) \to \Pi_1^*(\mathcal{F})_X$ be the homomorphism defined by $\eta(\alpha) = \alpha \otimes e$. Then η is surjective and we have the exact sequence

$$0 \to Ker(\eta) \to \Pi_1^*(\mathcal{F}) \xrightarrow{\eta} \Pi_1^*(\mathcal{F})_X \to 0$$

By theorem 3.5 of [8] the group $H^p(X \times \mathbb{C}^n, ker(\eta))$ vanishes for any $p \geq q + r$. Moreover, since $H^p(X \times \mathbb{C}^n, \Pi_1^*(\mathcal{F})) = 0$ for $p \geq q + r - 1$ and, clearly $\Pi_1^*(\mathcal{F})_X \cong \mathcal{F}$, the long exact sequence of cohomology

$$\cdots \to H^p(X \times \mathbb{C}^n, \Pi_1^*(\mathcal{F})) \to H^p(X, \Pi_1^*(\mathcal{F})_X) \to H^{p+1}(X \times \mathbb{C}^n, Ker(\eta)) \to \cdots$$

implies that

$$H^p(X, \mathcal{F}) \cong H^p(X, \Pi_1^*(\mathcal{F})_X) = 0 \text{ for all } p \geq q + r - 1.$$

■

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