Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 7, Number 4 (2011), pp. 435-438

http://www.ripublication.com/gjpam.htm

A not on Norms of Composition Operators Induced by Rational Symbol

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Abstract

Norms of composition operators with rational symbol that satisfy certain properties was studied by Sean Effinger-Dean, Alan Johnson, Joseph Reed, Janathan Shapiro in Norms of composition operators with rational symbol, J. Math. Anal. Appl. 324 (2006) 1062–1072.

In this article by using the methods of above paper we consider a self-map and study the norms of induced composition operator.

AMS Subject Classification:
Keywords:

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. The Hardy space H^2 is the familiar Hilbert space of analytic functions on $\mathbb{D}$ with square-summable Taylor coefficients.

For φ an analytic self-map of $\mathbb{D}$, C_φ denotes the composition operator defined by $C_\varphi f = f \circ \varphi$. *Littlewood's* Subordination Principle, which can be found in [7], guarantees that C_φ is a bounded operator on H^2.

One of the important problems in studying the composition operator C_φ, is calculating the norm of this operator. This is a difficult problem in general and there is a very limited collection of self-maps φ for which $\|C_\varphi\|$ is known exactly. It is well known that if φ is a inner function, then $\|C_\varphi\| = \sqrt{\dfrac{(1+|\varphi(0)|)}{(1-|\varphi(0)|)}}$, if φ is constant map $\varphi \equiv c$, then

$\|C_\varphi\| = \sqrt{\dfrac{1}{1-|c|^2}}$, and for all linear maps $\varphi(z) = sz + t$ with $|t| < 1$ and $|t| + |s| \leq 1$ (see [2], [3], p. 324), we have

$$\|C_\varphi\| = \sqrt{\frac{2}{1+|s|^2-|t|^2+\sqrt{(1-|s|^2+|t|^2)^2-4|t|^2}}}.$$

C. Hammond, in [5] and [6], and, with P. Bourdon, E. Fry, and C. Spofford in [1], developed techniques to compute the norm of a composition operator, in many cases, with linear fractional symbol.

In [4], by Sean Effinger-Dean, Alan Johnson, Joseph Reed, Janathan Shapiro, the methods of these earlier papers are extended and the composition operator norms are computed when the symbol φ is in a special class of rational functions.

2. Norms of composition operators induced by rational functions

For notational convenience, we introduce the following function:

Definition 2.1. $\rho : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ (where $\mathbb{C}^*$ denotes the extended complex plane) is defined by $\rho(z) = \dfrac{1}{\bar{z}}$. Note that $\rho^{-1} = \rho$ and for $z \in \partial\mathbb{D}$, $\rho(z) = z$.

The following theorems are proved in [4].

Theorem 2.2. Suppose $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ extends to a non-inner rational function on $\mathbb{C}^*$ and assume that C_φ is norm-attaining. Let $A = \{\zeta_k\}_{k=1}^n \subset \mathbb{D}$ denote the set of roots of the function $h(\zeta) = \zeta(1 - \varphi(0)\overline{(\varphi \circ \rho)})$. Suppose that each of these roots has multiplicity 1 and that $\varphi(A) \subset \{0, \varphi(0)\}$. Now let

$$a_1 = \sum_{\varphi(\zeta_k)=0} Res_{\zeta=\zeta_k} \frac{1}{\zeta(1-\varphi(0)\overline{(\varphi\circ\rho)(\zeta)})},$$

$$a_2 = \sum_{\varphi(\zeta_k)=\varphi(0)} Res_{\zeta=\zeta_k} \frac{1}{\zeta(1-\varphi(0)\overline{(\varphi\circ\rho)(\zeta)})}.$$

Then $\lambda = \|C_\varphi\|^2$ is the greatest solution to the following quadratic equation:

$$\lambda^2 - a_2\lambda - a_1 = 0.$$

Theorem 2.3. Suppose $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ extends to a non-inner rational function on $\mathbb{C}^*$ and assume that C_φ is norm-attaining. Let there exist non-empty sets $A = \{\zeta_i\}_{i=1}^m \subset \mathbb{D}$ and $B = \{z_j\}_{j=1}^n \subset \mathbb{D}$ with the following properties:

(1) Each root of $\zeta(1 - z_k\overline{(\varphi \circ \rho)(\zeta)})$ has multiplicity 1 and is an element of A.

(2) $\varphi(A) \subset B$.

Let M be the $n \times n$ matrix with entries

$$m_{jk} = \sum_{\varphi(\zeta_i)=z_j} Res_{\zeta=\zeta_i} \frac{1}{\zeta(1 - z_k\overline{(\varphi \circ \rho)(\zeta)})}.$$

Then $\|C_\varphi\|^2$ is the greatest eigenvalue of M.

The theorem 2.3, is the $n = 2$ version of the theorem **??**, with $B = \{0, \varphi(0)\}$. For $n \geq 3$, the linear fractional examples can be found, as in Hammond's work [[6], section 7], by using

$$\varphi(z) = \frac{(r-1)z - (n-1)}{-nz + r}$$

for $r > n$. The operator C_φ then satisfies the hypotheses of theorem **??**, with $B = \{\varphi(0), \tau(\varphi(0)), \tau(\tau(\varphi(0))), \cdots, \tau_{n-1}(\varphi(0)) = 0\}$, where $\tau(z) = \varphi(\sigma(z))$ and $\sigma(z) = \dfrac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$. In this section we give brief example for $n \geq 3$ version of the theorem **??**.

Example 2.4. Let $n = 3$ and $r = 4$, then $\varphi(z) = \dfrac{3z-2}{-3z+4}$ and $\varphi(0) = -\dfrac{1}{2}$, $\sigma(z) = \dfrac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}} = \dfrac{3z+3}{2z+4}$, $\sigma(\varphi(0)) = \dfrac{1}{2}$. Since $\tau(z) = \varphi(\sigma(z))$, we have $\tau(\varphi(0)) = \varphi(0)$, $\tau(\tau(\varphi(0))) = \varphi(\frac{2}{3}) = 0$, $B = \left\{-\dfrac{1}{2}, -\dfrac{1}{5}, 0\right\} \subset \mathbb{D}$ and $A = \left\{0, \dfrac{1}{2}, \dfrac{2}{3}\right\} \subset \mathbb{D}$. So each root of $\zeta(1 - z_k\overline{(\varphi \circ \rho)(\zeta)})$ has multiplicity 1 and is an element of A and $\varphi(A) \subset B$. Hence the hypotheses of theorem **??** satisfies.

Let M be the $n \times n$ matrix with entries m_{jk}. By using formula

$$m_{jk} = \sum_{\varphi(\zeta_i)=z_j} Res_{\zeta=\zeta_i} \frac{1}{\zeta(1 - z_k\overline{(\varphi \circ \rho)(\zeta)})}$$

It can be shown that M has the following form:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} 2 & \frac{5}{4} & 1 \\ -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{5}{36} & 0 \end{bmatrix}.$$

If $det(M - \lambda I) = 0$ where I is the identity matrix, $\lambda = \|C_\varphi\|^2$ is the greatest eigenvalue of M. From

$$\begin{vmatrix} 2-\lambda & \frac{5}{4} & 1 \\ -\frac{2}{3} & -\lambda & 0 \\ 0 & -\frac{5}{36} & -\lambda \end{vmatrix} = 0$$

we conclude that λ is the solution of the following equation

$$\lambda^3 - \lambda^2 + \frac{5}{6}\lambda - \frac{5}{54} = 0.$$

Hence $\lambda = \|C_\varphi\|^2$, the greatest solution to this equation is the greatest eigenvalue of M.

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