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Quantum Algorithm for Vertex Coloring Problem by Central Limit Theorem

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Abstract

A quantum algorithm for the vertex coloring problem by the central limit theorem and its example are reported. When n vertexes are connected m edges, and both vertexes of each edge are different colors, a number of colors that is k is decided. A computational complexity of a classical computation is k^n. The computational complexity becomes about n^2 by the quantum algorithm that uses quantum phase inversion gates, quantum inversion about mean gates, and the standard normal distribution. Therefore, a polynomial time process becomes possible.

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1. Introduction

A quantum computer can move quickly to resolve a problem by doing a parallel computation that uses quantum entangled states. Deutsch-Jozsa's algorithm for the rapid solution [1–3], Shor's algorithm for the factorization [2–4], Grover's algorithms for the database search [2,5,6], and so on are known. A quantum algorithm for the knapsack problem has recently been reported by Fujimura [7]. Its computational complexity becomes about square root for the computational complexity of a classical computation. In the vertex coloring problem [8] this time, a polynomial time process becomes possible by the central limit theorem. Therefore, its result is reported.

2. Vertex Coloring Problem

When n vertexes are connected by m edges, and both vertexes of each edge are different colors, a number of colors that is k is decided [8].

3. Quantum Algorithm

It is assumed that n vertexes are connected by m edges, and two vertexes have only one edge, because several edges don't change the essence of this problem. Therefore, when there is an edge between the i-th vertex and the j-th vertex, $x_{i,j}$ [$0 \leq i < j \leq n-1$. i and j are integers.] is 1, and when there isn't an edge between these vertexes, it is 0. Now, it is assumed that a number of colors is k. A distribution of $x_{i,j}$ becomes the following. When a random variable $X_{i,j}$ becomes $x_{i,j}$ as a probability $(k-1)/k$, a mean is $(k-1)(x_{i,j}/k)$ and a dispersion is $(k-1)(x_{i,j}/k)^2$. Therefore, when a total mean is $\mu = \sum_{0 \leq i < j \leq n-1} (k-1)(x_{i,j}/k) = (k-1)m/k$ and a total dispersion $\sigma^2 = \sum_{0 \leq i < j \leq n-1} (k-1)(x_{i,j}/k)^2 = (k-1)m/k^2$, $\left(\sum_{0 \leq i < j \leq n-1} X_{i,j} - \mu \right)/\sigma$ follows the normal distribution from the central limit theorem. When the standard normal distribution $f(z)$ [$\sigma = 1$] is $\int_0^z (e^{-z^2/2}/(2\pi)^{1/2})dz$, and values of $\int_z^{(m-\mu)/\sigma} (e^{-z^2/2}/(2\pi)^{1/2})dz$ are $1/2^2, 1/2^4, 1/2^6, 1/2^8, \cdots$, and $1/2^{2g}$ [g is a natural number], each value of z is assumed r_g. It is obtained from the table of $f(z)$. Each total number of the data between $\mu + r_g\sigma$ and m is $k^n/2^2, k^n/2^4, k^n/2^6, k^n/2^8, \cdots$, respectively.

Next, a quantum algorithm is shown as the following.

First of all, quantum registers $|a_0>, |a_1>, \cdots, |a_{n-1}>, |b>$, and $|c>$ are prepared. When α is a minimum integer that is $log_2 k$ or more, each of $|a_f>$ that f is an integer from 0 to $n-1$ is consisted of α quantum bits [= qubits]. States of $|a_0>, |a_1>, \cdots, |a_{n-1}>, |b>$, and $|c>$ are $a_0, a_1, \cdots, a_{n-1}, b$, and c, respectively.

Step 1: Each qubit of $|a_0>, |a_1>, \cdots, |a_{n-1}>, |b>$, and $|c>$ is set $|0>$.

Step 2: The Hadamard gate $\boxed{H}$ [2, 3] acts on each qubit of $|a_0>, |a_1>, \cdots, |a_{n-2}>$, and $|a_{n-1}>$. It changes them for entangled states. The total states are $(2^\alpha)^n$.

Step 3: It is assumed that a quantum gate (A) changes $|b>$ for $|b+1>$ at $a_f \geq k$, or it doesn't change $|b>$ at $a_f < k$. As a target state for $|b>$ is 0, quantum phase inversion gates (PI) and quantum inversion about mean gates (IM) [2,5,6] act on $|b>$. When β is a minimum even integer that is $(2^\alpha/k)^{1/2}$ or more, the total number that (PI) and (IM) act on $|b>$ is β because they are a couple. Next, an observation gate (OB) observes $|b>$. These actions are repeated sequentially from $|a_0>$ to $|a_{n-1}>$. Therefore, each state of $|a_f>$ is $0, 1, \cdots, k-2$, or $k-1$, and the total states become $k^n [= W_0]$.

Step 4: It is assumed that a quantum gate $(B_{i,j})$ [$0 \leq i < j \leq n-1$. i and j are integers.] changes $|b>$ for $|b + x_{i,j}>$ at $a_i \neq a_j$, or it doesn't change $|b>$ at $a_i = a_j$. This action repeats sequentially at i and j. Therefore, $|b>$ becomes from $|0>$ to $|x_{0,1} + x_{0,2} + \cdots + x_{n-2,n-1}>$.

Step 5: It is assumed that a quantum gate (C_1) doesn't change $|c>$ in $\mu + r_1\sigma \leq b \leq m$, or it changes $|c>$ for $|c+1>$ in the others of b. As the target state for $|c>$ is 0, (PI) and (IM) act on $|c>$. The number of the data that is included in $\mu + r_1\sigma \le b \le m$ is $W_1 \approx k^n/2^2$. When γ_1 is a minimum even integer that is $(W_0/W_1)^{1/2}$ or more, the total number that (PI) and (IM) act on $|c>$ is $\gamma_1 \approx 2$. Next, (OB) observes $|c>$, and the data of W_1 remain. Similarly, (C_v) [$2 \le v \le g-1$. v is an integer. g that is a minimum integer follows $k! \ge k^n/2^{2g}$. $k!$ are numbers at m at least.] doesn't change $|c>$ in $\mu + r_v\sigma \le b \le m$, or it changes $|c>$ for $|c+1>$ in the others of b. As the target state for $|c>$ is 0, (PI) and (IM) act on $|c>$. The number of the data that is included in $\mu + r_v\sigma \le b \le m$ is $W_v \approx k^n/2^{2v}$. When γ_v is the minimum even integer that is $(W_{v-1}/W_v)^{1/2}$ or more, the total number that (PI) and (IM) act on $|c>$ is $\gamma_v \approx 2$. Next, (OB) observes $|c>$, and the data of W_v remain. These actions are repeated sequentially from 2 to $g-1$ at v.

(C_g) doesn't change $|c>$ at $b=m$, or it changes $|c>$ for $|c+1>$ at $b \ne 0$. As the target state for $|c>$ is 0, (PI) and (IM) act on $|c>$. The number of the data that is included at $b=m$ is $W_g \approx k! \ge k^n/2^{2g}$. When γ_g is the minimum even integer that is $(W_{g-1}/W_g)^{1/2} \approx ((k^n/2^{2(g-1)})/(k^n/2^{2g}))^{1/2} = 2 \le 2 = \gamma_g$, the total number that (PI) and (IM) act on $|c>$ is $\gamma_g \approx 2$. Next, (OB) observes $|a_0>, |a_1>, \cdots, |a_{n-1}>, |b>$, and $|c>$, and one of the data of W_g remain. When c is 0, a sum of $x_{i,j}$ becomes $m[=b]$.

4. Numerical Computation

It is assumed that there are $n=4, x_{0,1}=x_{0,2}=x_{0,3}=x_{1,2}=x_{2,3}=1, x_{1,3}=0, m=5, k=3, \mu=10/3 \approx 3.333, \sigma=(10/9)^{1/2} \approx 1.054, g=2, r_1=0.5046$, and $r_2=1.178$.

First of all, $|a_0>, |a_1>, |a_2>, |a_3>, |b>$, and $|c>$ are prepared. When α is the minimum integer that is $log_2 3 \approx 1.6 \le 2 = \alpha$, each of $|a_f>$ that f is the integer from 0 to 3 is consisted of 2 qubits. States of $|a_0>, |a_1>, |a_2>, |a_3>, |b>$, and $|c>$ are a_0, a_1, a_2, a_3, b, and c, respectively.

Step 1: Each qubit of $|a_0>, |a_1>, |a_2>, |a_3>, |b>$, and $|c>$ is set $|0>$.

Step 2: $\boxed{\text{H}}$ acts on each qubit of $|a_0>, |a_1>, |a_2>$, and $|a_3>$. It changes them for entangled states. The total states are $(2^2)^4$.

Step 3: (A) changes $|b>$ for $|b+1>$ at $a_f \ge 3$, or it doesn't change $|b>$ at $a_f < 3$. As the target state for $|b>$ is 0, (PI) and (IM) act on $|b>$. When β is the minimum even integer that is $(2^2/3)^{1/2} \approx 1.2 \le 2 = \beta$, the total number that (PI) and (IM) act on $|b>$ is $\beta = 2$. Next, (OB) observes $|b>$. These actions are repeated sequentially from $|a_0>$ to $|a_3>$. Therefore, each state of $|a_f>$ is 0, 1, or 2, and the total states become $3^4[=W_0]$.

Step 4: $(B_{i,j})$ [$0 \le i < j \le 3$] changes $|b>$ for $|b+x_{i,j}>$ at $a_i \ne a_j$, or it doesn't change $|b>$ at $a_i = a_j$. This action repeats sequentially at i and j. Therefore, $|b>$ becomes from $|0>$ to $|x_{0,1}+x_{0,2}+\cdots+x_{2,3}>$.

Step 5: (C_1) doesn't change $|c>$ in $\mu + r_1\sigma \approx 3.3 + 0.5 = 3.8 \leq b \leq m = 5$, or it changes $|c>$ for $|c+1>$ in the others of b. As the target state for $|c>$ is 0, (PI) and (IM) act on $|c>$. The number of the data that is included in $3.8 \leq b \leq 5$ is $W_1 \approx 3^4/2^2$. When γ_1 is a minimum even integer that is $(W_0/W_1)^{1/2} \approx (3^4/(3^4/2^2))^{1/2} = 2 \leq 2 = \gamma_1$, the total number that (PI) and (IM) act on $|c>$ is $\gamma_1 \approx 2$. Next, (OB) observes $|c>$, and the data of W_1 remain.

(C_2) doesn't change $|c>$ at $\mu + r_2\sigma \approx 3.3+1.2 = 4.5 \leq b \leq 5[b = 5]$, or it changes $|c>$ for $|c+1>$ at $b \neq 5$. As the target state for $|c>$ is 0, (PI) and (IM) act on $|c>$. The number of the data that is included at $b = 5$ is $W_2 \approx k! = 3! \geq 3^4/2^4$. When γ_2 is the minimum even integer that is $(W_1/W_2)^{1/2} \approx ((3^4/2^2)/(3^4/2^4))^{1/2} = 2 \leq 2 = \gamma_2$, the total number that (PI) and (IM) act on $|c>$ is $\gamma_2 \approx 2$. Next, (OB) observes $|a_0>, |a_1>, |a_2>, |a_3>, |b>$, and $|c>$, and one of the data of W_2 remain. For example, when a_0, a_1, a_2, a_3, b, and c are 0, 2, 1, 0, 5, and 0, respectively. it is obtained that the 0th and 3rd vertexes are the 0th color, the 2nd vertex is the 1st color, and the 1st vertex is the 2nd color. Therefore, $3! = 6$ combinations are obtained from this answer at least.

5. Discussion and Summary

The computational complexity of this quantum algorithm [$= S$] becomes the following. In the order of the actions by the gates, the number of them is αn at $\boxed{\text{H}}$, n at (A), $\beta n = 2n$ at (PI) and (IM), n at (OB), $n(n-1)$ at $(B_{i,j})[0 \leq i < j \leq n-1]$, g at $(C_v)[1 \leq v \leq g]$, $\sum_{v=1}^{g} \gamma_v = 2g$ at (PI) and (IM), and g at (OB). Therefore, S becomes $n^2 + (\alpha+3)n + 4g$. In the example of the section 4, S is 32. The computational complexity of the classical computation [$= Z$] is $k^n = 3^4 = 81$. After all, S/Z becomes about 1/2. When n is large enough, S becomes about $n^2 + (\alpha + 3)n + 4g \leq n^2 + (log_2k + 3)n + 4 \times (n/2)log_2k = n^2 + 3(log_2k + 1)n \approx n^2$, where a maximum value of g is about $(n/2)log_2k$, and S/Z is about n^2/k^n. For example, as for $n = 100$ and $k = 4$, S/Z is about $100^2/4^{100} \approx 10^4/10^{60} = 1/10^{56}$.

Therefore, a polynomial time process becomes possible.

References

[1] Deutsch D., and Jozsa R., Rapid solution of problems by quantum computation, *Proc. Roy. Soc. Lond. A*, 439:553–558, 1992.

[2] Takeuchi S., Ryoshi Konpyuta (Quantum Computer), Kodansha, Tokyo, Japan, 2005 [in Japanese].

[3] Miyano K., and Furusawa A., Ryoshi Konpyuta Nyumon (An Introduction to Quantum Computation), Nihonhyoronsha, Tokyo, Japan, 2008 [in Japanese].

[4] Shor P.W., Algorithms for quantum computation: discrete logarithms and factoring, *Proc. 35th Annu. Symp. Foundations of Computer Science*, IEEE, pp. 124–134, 1994.

[5] Grover L.K., A fast quantum mechanical algorithm for database search, *Proc. 28th Annu. ACM Symp. Theory of Computing*, pp. 212–219, 1996.

[6] Grover L.K., A framework for fast quantum mechanical algorithms, *Proc. 30th Annu. ACM Symp. Theory of Computing*, pp. 53–62, 1998.

[7] Fujimura T., Quantum algorithm for knapsack problem, *Glob. J. Pure Appl. Math.*, 6:263–266, 2010.

[8] Weisstein E.W., Vertex coloring, [Online], 2011. Available: http://mathworld.wolfram.com/VertexColoring.html.