

A Fixed Point Theorem In Intuitionistic \mathcal{M} -fuzzy Metric Spaces

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Abstract

In this paper, we define the notion of intuitionistic \mathcal{M} -fuzzy metric spaces (see, [13]) as a natural generalization of intuitionistic fuzzy metric space and \mathcal{M} -fuzzy metric space. Also, we introduce the notion of Cauchy sequence in an intuitionistic \mathcal{M} -fuzzy metric spaces. Further, we prove the well-known fixed point theorem of Banach is extended to intuitionistic \mathcal{M} -fuzzy metric spaces.

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1. Introduction and Preliminaries

The theory of fuzzy sets proposed by Zadeh [14] has showed successful applications in various fields and laid the foundation of fuzzy mathematics. Especially, several mathematicians have introduced fuzzy metric spaces in different ways (see for example [2], [4], [5], [7], [8]). In 1983, Atanassov [1] has introduced the notion of intuitionistic fuzzy sets and Park [9] has defined intuitionistic fuzzy metric spaces and obtained several classical theorems on this new structure. Recently Sedghi et al. [12] introduced D^* -metric space as a probable modification of the definition of D -metric introduced by Dhage [3]. Using D^* -metric concepts, They [11] defined \mathcal{M} -fuzzy metric space and proved a common fixed point theorem in it.

We [13] given some new definitions of intuitionistic \mathcal{M} -fuzzy metric spaces with the help of continuous t -norms and continuous t -conorms as a generalization of \mathcal{M} -fuzzy metric space.

In this paper first, we state some known definitions and results in an intuitionistic \mathcal{M} -fuzzy metric space. Then, we introduce the notion of Cauchy sequence in an intuitionistic \mathcal{M} -fuzzy metric spaces. Further, we prove the well-known fixed point theorem of Banach is extended to intuitionistic \mathcal{M} -fuzzy metric spaces. Our result generalizes and extends many known results in \mathcal{M} -fuzzy metric spaces and intuitionistic fuzzy metric spaces.

In what follows, \mathbf{N} the set of all natural numbers and \mathbf{R}^+ the set of all positive real numbers.

Definition 1.1. [10] A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t -norm if $*$ is satisfying the following conditions:

- (1) $*$ is commutative and associative;
- (2) $*$ is continuous;
- (3) $a * 1 = a$ for all $a \in [0, 1]$;
- (4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Two typical examples of continuous t -norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2. [10] A binary operation \diamond : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t -conorm if \diamond is satisfying the following conditions:

- (1) \diamond is commutative and associative;
- (2) \diamond is continuous;
- (3) $a \diamond 1 = a$ for all $a \in [0, 1]$;
- (4) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Two typical examples of continuous t -conorm are $a \diamond b = \max\{a, b\}$ and $a \diamond b = \min\{1, a + b\}$.

Definition 1.3. A 5-tuple $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is called an intuitionistic \mathcal{M} -fuzzy metric space if X is an arbitrary set, $*$ is a continuous t -norm, \diamond is a continuous t -conorm and \mathcal{M}, \mathcal{N} are fuzzy sets on $X^3 \times (0, \infty)$ satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$,

- (1) $\mathcal{M}(x, y, z, t) + \mathcal{N}(x, y, z, t) \leq 1$;
- (2) $\mathcal{M}(x, y, z, t) > 0$;
- (3) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$;
- (4) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, (symmetry) where p is a permutation function;
- (5) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$;

- (6) $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
- (7) $\mathcal{N}(x, y, z, t) > 0$;
- (8) $\mathcal{N}(x, y, z, t) = 0$ if and only if $x = y = z$;
- (9) $\mathcal{N}(x, y, z, t) = \mathcal{N}(p\{x, y, z\}, t)$, (symmetry) where p is a permutation function;
- (10) $\mathcal{N}(x, y, a, t) \diamond \mathcal{N}(a, z, z, s) \geq \mathcal{N}(x, y, z, t + s)$;
- (11) $\mathcal{N}(x, y, z, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Then $(\mathcal{M}, \mathcal{N})$ is called an intuitionistic \mathcal{M} -fuzzy metric on X . The functions $\mathcal{M}(x, y, z, t)$ and $\mathcal{N}(x, y, z, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t , respectively.

Remark 1.4. Since $*$ and \diamond are continuous, the limit is uniquely determined from (5) and (10). Throughout this paper, $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ will denote the intuitionistic \mathcal{M} -fuzzy metric space in the sense of Definition ?? with the following conditions:

- (12) $\lim_{t \rightarrow \infty} \mathcal{M}(x, y, z, t) = 1$ for all $x, y, z \in X$ and $t > 0$;
- (13) $\lim_{t \rightarrow \infty} \mathcal{N}(x, y, z, t) = 0$ for all $x, y, z \in X$ and $t > 0$.

Lemma 1.5. Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic \mathcal{M} -fuzzy metric space. For any $x, y \in X$ and $t > 0$, we have

- (1) $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.
- (2) $\mathcal{N}(x, x, y, t) = \mathcal{N}(x, y, y, t)$.

Proof. (1) Because for each $\epsilon > 0$ by triangular inequality, we have

- (1.1) $\mathcal{M}(x, x, y, \epsilon + t) \geq \mathcal{M}(x, x, x, \epsilon) * \mathcal{M}(x, y, y, t) = \mathcal{M}(x, y, y, t)$,
- (1.2) $\mathcal{M}(y, y, x, \epsilon + t) \geq \mathcal{M}(y, y, y, \epsilon) * \mathcal{M}(y, x, x, t) = \mathcal{M}(y, x, x, t)$.

By taking limit $\epsilon \rightarrow 0$ in (1.1) and (1.2), we get

$$\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t).$$

(2) Because for each $\epsilon > 0$ by triangular inequality, we have

- (2.1) $\mathcal{N}(x, x, y, \epsilon + t) \leq \mathcal{N}(x, x, x, \epsilon) \diamond \mathcal{N}(x, y, y, t) = \mathcal{N}(x, y, y, t)$,
- (2.2) $\mathcal{N}(y, y, x, \epsilon + t) \leq \mathcal{N}(y, y, y, \epsilon) \diamond \mathcal{N}(y, x, x, t) = \mathcal{N}(y, x, x, t)$.

By taking limit $\epsilon \rightarrow 0$ in (2.1) and (2.2), we get

$$\mathcal{N}(x, x, y, t) = \mathcal{N}(x, y, y, t).$$

This completes the proof. ■

Example 1.6. [9] Let X be a nonempty set with D^* -metric on X . Denote $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, Define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)} \quad \text{and} \quad \mathcal{N}(x, y, z, t) = \frac{D^*(x, y, z)}{t + D^*(x, y, z)}$$

for all $x, y, z \in X$. It is easy to see that $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is an intuitionistic \mathcal{M} -fuzzy metric space.

Definition 1.7. Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic \mathcal{M} -fuzzy metric space, For $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_{\mathcal{M}}(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r, \mathcal{N}(x, y, y, t) < r\}.$$

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$, such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.

Definition 1.8. Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic \mathcal{M} -fuzzy metric space and $\{x_n\}$ be a sequence in X .

(1) $\{x_n\}$ is said to be convergent to a point $x \in X$ (denoted by $\lim_{n \rightarrow \infty} x_n = x$) if $\lim_{n \rightarrow \infty} \mathcal{M}(x, x, x_n, t) = 1$ and $\lim_{n \rightarrow \infty} \mathcal{N}(x, x, x_n, t) = 0$ for all $t > 0$.

(2) $\{x_n\}$ is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exist $n_0 \in \mathbf{N}$, such that

$$\mathcal{M}(x_n, x_n, x_m, t) > 1 - \epsilon \quad \text{and} \quad \mathcal{N}(x_n, x_n, x_m, t) < \epsilon$$

for all $n, m \geq n_0$.

(3) An intuitionistic \mathcal{M} -fuzzy metric in which every Cauchy sequence is convergent is said to be complete.

Lemma 1.9. Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic \mathcal{M} -fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is non-decreasing and $\mathcal{N}(x, y, z, t)$ is non-increasing with respect to t , for all $x, y, z \in X$.

Proof. By Definition ?? for each $x, y, z, a \in X$ and $t, s > 0$, we have

$$\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$$

and

$$\mathcal{N}(x, y, a, t) \diamond \mathcal{N}(a, z, z, s) \geq \mathcal{N}(x, y, z, t + s).$$

If set $a = z$ we get

$$\mathcal{M}(x, y, z, t) * \mathcal{M}(z, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$$

and

$$\mathcal{N}(x, y, z, t) \diamond \mathcal{N}(z, z, z, s) \geq \mathcal{N}(x, y, z, t + s),$$

that is,

$$\mathcal{M}(x, y, z, t + s) \geq \mathcal{M}(x, y, z, t)$$

and

$$\mathcal{N}(x, y, z, t + s) \leq \mathcal{N}(x, y, z, t).$$

This completes the proof. ■

Definition 1.10. Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic \mathcal{M} -fuzzy metric space. \mathcal{M} and \mathcal{N} are said to be continuous functions on $X^3 \times (0, \infty)$ if

$$\lim_{n \rightarrow \infty} \mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t) \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}(x_n, y_n, z_n, t_n) = \mathcal{N}(x, y, z, t)$$

whenever a sequence $\{(x_n, y_n, z_n, t_n)\}$ in $X^3 \times (0, \infty)$ converges to a point $(x, y, z, t) \in X^3 \times (0, \infty)$, i.e.,

$$\lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} y_n = y, \lim_{n \rightarrow \infty} z_n = z,$$

$$\lim_{n \rightarrow \infty} \mathcal{M}(x, y, z, t_n) = \mathcal{M}(x, y, z, t) \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}(x, y, z, t_n) = \mathcal{N}(x, y, z, t).$$

Lemma 1.11. [13] Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic \mathcal{M} -fuzzy metric space. Then \mathcal{M} and \mathcal{N} are continuous functions on $X^3 \times (0, \infty)$.

2. Main Results

Now, we present the main result in this paper. Hence forth, we assume that $*$ is a continuous t -norm and \diamond is a continuous t -conorm on $[0, 1]$, such that for every $\mu \in (0, 1)$, there is a $\lambda \in (0, 1)$, such that

$$\underbrace{(1 - \lambda) * (1 - \lambda) * \dots * (1 - \lambda)}_n \geq 1 - \mu \text{ and } \underbrace{\lambda \diamond \lambda \diamond \dots \diamond \lambda}_n \leq \mu.$$

Lemma 2.1. Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic \mathcal{M} -fuzzy metric space. If we define $E_{\lambda, \mathcal{M}} : X^3 \rightarrow \mathbf{R}^+ \cup \{0\}$ by

$$E_{\lambda, \mathcal{M}}(x, y, z) = \inf\{t > 0 : \mathcal{M}(x, y, z, t) > 1 - \lambda\}$$

and $E_{\lambda, \mathcal{N}} : X^3 \rightarrow \mathbf{R}^+ \cup \{0\}$ by

$$E_{\lambda, \mathcal{N}}(x, y, z) = \sup\{t > 0 : \mathcal{N}(x, y, z, t) < \lambda\}$$

for all $\lambda \in (0, 1)$ and $x, y, z \in X$, then

(1) For all $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$, such that

$$E_{\mu, \mathcal{M}}(x_1, x_1, x_n) \leq E_{\lambda, \mathcal{M}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{M}}(x_2, x_2, x_3) + \cdots \\ + E_{\lambda, \mathcal{M}}(x_{n-1}, x_{n-1}, x_n)$$

and

$$E_{\mu, \mathcal{N}}(x_1, x_1, x_n) \geq E_{\lambda, \mathcal{N}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{N}}(x_2, x_2, x_3) + \cdots \\ + E_{\lambda, \mathcal{N}}(x_{n-1}, x_{n-1}, x_n)$$

for all $x_1, x_2, \dots, x_n \in X$.

(2) The sequence $\{x_n\}_{n \in \mathbb{N}}$ is convergent in intuitionistic \mathcal{M} -fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ if and only if $E_{\lambda, \mathcal{M}}(x_n, x_n, x) \rightarrow 0$ and $E_{\lambda, \mathcal{N}}(x_n, x_n, x) \rightarrow 0$. Also the sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if and only if it is a Cauchy sequence with $E_{\lambda, \mathcal{M}}$ and $E_{\lambda, \mathcal{N}}$.

Proof. (1) For every $\mu \in (0, 1)$, we can find a $\lambda \in (0, 1)$, such that

$$\underbrace{(1 - \lambda) * (1 - \lambda) * \cdots * (1 - \lambda)}_n \geq 1 - \mu \quad \text{and} \quad \underbrace{\lambda \diamond \lambda \diamond \cdots \diamond \lambda}_n \leq \mu.$$

So by triangle inequality, we have $\mathcal{M}(x_1, x_1, x_n, E_{\lambda, \mathcal{M}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{M}}(x_2, x_2, x_3) + \cdots + E_{\lambda, \mathcal{M}}(x_{n-1}, x_{n-1}, x_n) + n\delta)$

$$\geq \mathcal{M}(x_1, x_1, x_2, E_{\lambda, \mathcal{M}}(x_1, x_1, x_2) + \delta) * \cdots * \mathcal{M}(x_{n-1}, x_{n-1}, x_n, \\ E_{\lambda, \mathcal{M}}(x_{n-1}, x_{n-1}, x_n) + \delta) \\ \geq \underbrace{(1 - \lambda) * (1 - \lambda) * \cdots * (1 - \lambda)}_n \geq 1 - \mu$$

and

$$\mathcal{N}(x_1, x_1, x_n, E_{\lambda, \mathcal{N}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{N}}(x_2, x_2, x_3) + \cdots + E_{\lambda, \mathcal{N}}(x_{n-1}, x_{n-1}, x_n) - n\delta) \\ \leq \mathcal{N}(x_1, x_1, x_2, E_{\lambda, \mathcal{N}}(x_1, x_1, x_2) - \delta) \diamond \cdots \diamond \mathcal{N}(x_{n-1}, x_{n-1}, x_n, \\ E_{\lambda, \mathcal{N}}(x_{n-1}, x_{n-1}, x_n) - \delta) \\ \leq \underbrace{\lambda \diamond \lambda \diamond \cdots \diamond \lambda}_n \leq \mu$$

for all $\delta > 0$, which implies that

$$E_{\mu, \mathcal{M}}(x_1, x_1, x_n) \leq E_{\lambda, \mathcal{M}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{M}}(x_2, x_2, x_3) + \cdots \\ + E_{\lambda, \mathcal{M}}(x_{n-1}, x_{n-1}, x_n) + n\delta$$

and

$$E_{\mu, \mathcal{N}}(x_1, x_1, x_n) \geq E_{\lambda, \mathcal{N}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{N}}(x_2, x_2, x_3) + \cdots + E_{\lambda, \mathcal{N}}(x_{n-1}, x_{n-1}, x_n) - n\delta.$$

Since $\delta > 0$ is arbitrary, we have

$$E_{\mu, \mathcal{M}}(x_1, x_1, x_n) \leq E_{\lambda, \mathcal{M}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{M}}(x_2, x_2, x_3) + \cdots + E_{\lambda, \mathcal{M}}(x_{n-1}, x_{n-1}, x_n)$$

and

$$E_{\mu, \mathcal{N}}(x_1, x_1, x_n) \geq E_{\lambda, \mathcal{N}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{N}}(x_2, x_2, x_3) + \cdots + E_{\lambda, \mathcal{N}}(x_{n-1}, x_{n-1}, x_n).$$

For (2), Note that since \mathcal{M} and \mathcal{N} are continuous in its third place and

$$E_{\lambda, \mathcal{M}}(x, x, y) = \inf\{t > 0 : \mathcal{M}(x, x, y, t) > 1 - \lambda\}$$

and

$$E_{\lambda, \mathcal{N}}(x, x, y) = \sup\{t > 0 : \mathcal{N}(x, x, y, t) < \lambda\}.$$

Hence, we have

$$\mathcal{M}(x_n, x_n, x, \eta) > 1 - \lambda \iff E_{\lambda, \mathcal{M}}(x_n, x_n, x) < \eta$$

and

$$\mathcal{N}(x_n, x_n, x, \eta) < \lambda \iff E_{\lambda, \mathcal{N}}(x_n, x_n, x) > \eta$$

for all $\eta > 0$. This completes the proof. ■

Theorem 2.2. Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be an intuitionistic \mathcal{M} -fuzzy metric space. If

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) \geq \mathcal{M}(x_0, x_0, x_1, k^n t) \text{ and } \mathcal{N}(x_n, x_n, x_{n+1}, t) \leq \mathcal{N}(x_0, x_0, x_1, k^n t)$$

for some $k > 1$ and for every $n \in \mathbf{N}$. Then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For every $\lambda \in (0, 1)$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned} E_{\lambda, \mathcal{M}}(x_n, x_n, x_{n+1}) &= \inf\{t > 0 : \mathcal{M}(x_n, x_n, x_{n+1}, t) > 1 - \lambda\} \\ &\leq \inf\{t > 0 : \mathcal{M}(x_0, x_0, x_1, k^n t) > 1 - \lambda\} \\ &= \inf\{\frac{t}{k^n} > 0 : \mathcal{M}(x_0, x_0, x_1, t) > 1 - \lambda\} \\ &= \frac{1}{k^n} \inf\{t > 0 : \mathcal{M}(x_0, x_0, x_1, t) > 1 - \lambda\} \\ &= \frac{1}{k^n} E_{\lambda, \mathcal{M}}(x_0, x_0, x_1) \end{aligned}$$

and

$$\begin{aligned} E_{\lambda, \mathcal{N}}(x_n, x_n, x_{n+1}) &= \sup\{t > 0 : \mathcal{N}(x_n, x_n, x_{n+1}, t) < \lambda\} \\ &\geq \sup\{t > 0 : \mathcal{N}(x_0, x_0, x_1, k^n t) < \lambda\} \\ &= \sup\{\frac{t}{k^n} > 0 : \mathcal{N}(x_0, x_0, x_1, t) < \lambda\} \\ &= \frac{1}{k^n} \sup\{t > 0 : \mathcal{N}(x_0, x_0, x_1, t) < \lambda\} \\ &= \frac{1}{k^n} E_{\lambda, \mathcal{N}}(x_0, x_0, x_1). \end{aligned}$$

By Lemma ??, for every $\mu \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that $E_{\mu, \mathcal{M}}(x_n, x_n, x_m)$

$$\begin{aligned} &\leq E_{\lambda, \mathcal{M}}(x_n, x_n, x_{n+1}) + E_{\lambda, \mathcal{M}}(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + E_{\lambda, \mathcal{M}}(x_{m-1}, x_{m-1}, x_m) \\ &\leq \frac{1}{k^n} E_{\lambda, \mathcal{M}}(x_0, x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda, \mathcal{M}}(x_0, x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda, \mathcal{M}}(x_0, x_0, x_1) \\ &= E_{\lambda, \mathcal{M}}(x_0, x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \rightarrow 0 \end{aligned}$$

and $E_{\mu, \mathcal{N}}(x_n, x_n, x_m)$

$$\begin{aligned} &\geq E_{\lambda, \mathcal{N}}(x_n, x_n, x_{n+1}) + E_{\lambda, \mathcal{N}}(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + E_{\lambda, \mathcal{N}}(x_{m-1}, x_{m-1}, x_m) \\ &\geq \frac{1}{k^n} E_{\lambda, \mathcal{N}}(x_0, x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda, \mathcal{N}}(x_0, x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda, \mathcal{N}}(x_0, x_0, x_1) \\ &= E_{\lambda, \mathcal{N}}(x_0, x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \rightarrow 0. \end{aligned}$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence. This completes the proof. \blacksquare

The following theorem is Jungck's [6] generalization of Banach's contraction principle in metric spaces.

Theorem 2.3. Let f be a continuous mapping of a complete metric space (X, d) into itself and let $g : X \rightarrow X$ be a map that satisfy the following conditions:

- (a) $g(X) \subseteq f(X)$;
- (b) g commutes with f ;
- (c) $d(g(x), g(y)) \leq kd(f(x), f(y))$ for all $x, y \in X$ and for some $0 < k < 1$.

Then f and g have a unique common fixed point.

The above result has an intuitionistic \mathcal{M} -fuzzy metric space analogue in the following theorem.

Theorem 2.4. Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be a complete intuitionistic \mathcal{M} -fuzzy metric space and let $f, g : X \rightarrow X$ be maps that satisfy the following conditions:

- (a) $g(X) \subseteq f(X)$;
- (b) f is continuous;
- (c) $\mathcal{M}(g(x), g(y), g(y), kt) \geq \mathcal{M}(f(x), f(y), f(y), t)$ and $\mathcal{N}(g(x), g(y), g(y), kt) \leq \mathcal{N}(f(x), f(y), f(y), t)$;

for all $x, y \in X$ and for some $0 < k < 1$.

Then f and g have a unique common fixed point provided f and g commute.

Proof. Let $x_0 \in X$. By (a), we can find x_1 such that $f(x_1) = g(x_0)$. By induction, we can define a sequence $\{x_n\}$ such that $f(x_n) = g(x_{n-1})$. By induction again,

$$\begin{aligned} \mathcal{M}(f(x_n), f(x_{n+1}), f(x_{n+1}), t) &= \mathcal{M}(g(x_{n-1}), g(x_n), g(x_n), t) \\ &\geq \mathcal{M}(f(x_{n-1}), f(x_n), f(x_n), \frac{t}{k}) \\ &\geq \dots \geq \mathcal{M}\left(f(x_0), f(x_1), f(x_1), \frac{t}{k^n}\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}(f(x_n), f(x_{n+1}), f(x_{n+1}), t) &= \mathcal{N}(g(x_{n-1}), g(x_n), g(x_n), t) \\ &\leq \mathcal{N}\left(f(x_{n-1}), f(x_n), f(x_n), \frac{t}{k}\right) \\ &\leq \dots \leq \mathcal{N}\left(f(x_0), f(x_1), f(x_1), \frac{t}{k^n}\right) \end{aligned}$$

for $n = 1, 2, \dots$, which implies that

$$\begin{aligned} E_{\lambda, \mathcal{M}}(f(x_n), f(x_{n+1}), f(x_{n+1})) &= \inf\{t > 0 : \mathcal{M}(f(x_n), f(x_{n+1}), f(x_{n+1}), t) > 1 - \lambda\} \\ &\leq \inf\{t > 0 : \mathcal{M}\left(f(x_0), f(x_1), f(x_1), \frac{t}{k^n}\right) > 1 - \lambda\} \\ &= k^n \inf\{t > 0 : \mathcal{M}(f(x_0), f(x_1), f(x_1), t) > 1 - \lambda\} \\ &= k^n E_{\lambda, \mathcal{M}}(f(x_0), f(x_1), f(x_1)) \end{aligned}$$

and

$$\begin{aligned} E_{\lambda, \mathcal{N}}(f(x_n), f(x_{n+1}), f(x_{n+1})) &= \sup\{t > 0 : \mathcal{N}(f(x_n), f(x_{n+1}), f(x_{n+1}), t) < \lambda\} \\ &\geq \sup\{t > 0 : \mathcal{N}\left(f(x_0), f(x_1), f(x_1), \frac{t}{k^n}\right) < \lambda\} \\ &= k^n \sup\{t > 0 : \mathcal{N}(f(x_0), f(x_1), f(x_1), t) < \lambda\} \\ &= k^n E_{\lambda, \mathcal{N}}(f(x_0), f(x_1), f(x_1)). \end{aligned}$$

for every $\lambda \in (0, 1)$. Now, we show that $\{f(x_n)\}$ is a left Cauchy sequence. For every $\mu \in (0, 1)$, there exists $\gamma \in (0, 1)$ such that, for $m \geq n$,

$$\begin{aligned} E_{\mu, \mathcal{M}}(f(x_n), f(x_m), f(x_m)) &\leq E_{\gamma, \mathcal{M}}(f(x_{m-1}), f(x_m), f(x_m)) \\ &\quad + E_{\gamma, \mathcal{M}}(f(x_{m-2}), f(x_{m-1}), f(x_{m-1})) \\ &\quad + \dots + E_{\gamma, \mathcal{M}}(f(x_n), f(x_{n+1}), f(x_{n+1})) \\ &\leq E_{\gamma, \mathcal{M}}(f(x_0), f(x_1), f(x_1)) \sum_{j=n}^{m-1} k^j \rightarrow 0. \end{aligned}$$

and

$$\begin{aligned} E_{\mu, \mathcal{N}}(f(x_n), f(x_m), f(x_m)) &\geq E_{\gamma, \mathcal{N}}(f(x_{m-1}), f(x_m), f(x_m)) \\ &\quad + E_{\gamma, \mathcal{N}}(f(x_{m-2}), f(x_{m-1}), f(x_{m-1})) \\ &\quad + \dots + E_{\gamma, \mathcal{N}}(f(x_n), f(x_{n+1}), f(x_{n+1})) \\ &\geq E_{\gamma, \mathcal{N}}(f(x_0), f(x_1), f(x_1)) \sum_{j=n}^{m-1} k^j \rightarrow 0. \end{aligned}$$

as $m, n \rightarrow \infty$. Since the space X is left complete, there exists a point $y \in X$ such that $\lim_{n \rightarrow \infty} f(x_n) = y$. So $g(x_{n-1}) = f(x_n)$ tends to y . It can be seen from (c) that the continuity of f implies the continuity of g . Therefore, $\{g(f(x_n))\}$ converges to $g(y)$. However, $g(f(x_n)) = f(g(x_n))$ by the commutativity of f and g . So $\{f(g(x_n))\}$ converges to $f(y)$. Because the limits are unique, $f(y) = g(y)$. So $f(f(y)) = f(g(y))$ by commutativity,

$$\begin{aligned} \mathcal{M}(g(y), g(g(y)), g(g(y)), t) &\geq \mathcal{M}\left(f(y), f(g(y)), f(g(y)), \frac{t}{k}\right) \\ &\geq \mathcal{M}\left(g(y), g(g(y)), g(g(y)), \frac{t}{k}\right) \\ &\geq \dots \geq \mathcal{M}\left(g(y), g(g(y)), g(g(y)), \frac{t}{k^n}\right) \rightarrow 1 \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}(g(y), g(g(y)), g(g(y)), t) &\leq \mathcal{N}\left(f(y), f(g(y)), f(g(y)), \frac{t}{k}\right) \\ &\leq \mathcal{N}\left(g(y), g(g(y)), g(g(y)), \frac{t}{k}\right) \\ &\leq \dots \leq \mathcal{N}\left(g(y), g(g(y)), g(g(y)), \frac{t}{k^n}\right) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, which implies that, $g(y) = g(g(y))$. Thus $g(y) = g(g(y)) = f(g(y))$, which implies that $g(y)$ is a common fixed point of f and g . If y and z are two common fixed points of f and g , then

$$\begin{aligned} 1 \geq \mathcal{M}(y, z, z, t) = \mathcal{M}(g(y), g(z), g(z), t) &\geq \mathcal{M}\left(f(y), f(z), f(z), \frac{t}{k}\right) \\ &= \mathcal{M}\left(y, z, z, \frac{t}{k}\right) \geq \dots \\ &\geq \mathcal{M}\left(y, z, z, \frac{t}{k^n}\right) \rightarrow 1 \end{aligned}$$

and

$$\begin{aligned} 0 \leq \mathcal{N}(y, z, z, t) = \mathcal{N}(g(y), g(z), g(z), t) &\leq \mathcal{N}\left(f(y), f(z), f(z), \frac{t}{k}\right) \\ &= \mathcal{N}\left(y, z, z, \frac{t}{k}\right) \leq \dots \\ &\leq \mathcal{N}\left(y, z, z, \frac{t}{k^n}\right) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, which implies that $y = z$ and so the uniqueness of the common fixed point follows. ■

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