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Estimating Dimension and Codimension by Polynomials on *T*

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Abstract

In this paper we consider the space of the functions having finitely many singularity points and the closure for the space of the trigonometric polynomials intersecting its conjugate on unit circle. We use polynomials with coefficients on the unit circle to estimate the dimension for their closure and estimate the codimension for their generating subspace.

Keywords: Dimension, Polynomials.

Introduction

Throughout this paper R is the additive group of real numbers and Z is the subgroup consisting of integers. T is the quotient group $R/2\pi Z$. Any function on T can be identify by a 2π-periodic function on R. A function f is integrable on T if its corresponding 2π-periodic function is integrable on $[0,2\pi)$ and we consider this interval as a model for T and the Lebesgue measure dt on T. Therefore, by $\int_T f(t)dt$ we mean $\int_0^{2\pi} f(x)dx$. For $1 \le p \le \infty$ let $L^p(\mu)$ be the space of complex-valued measurable functions on T with finite usual norm. That is, if $f \in L^p(\mu)$, then $\| f \|_p^p = \int_T | f(t)|^p \ \mu(t)dt$ whenever $1 \le p < \infty$ and $\| f \|_p = ess\sup_T | f(t) | \mu(t)$ if $p = \infty$. For more details, see [1], [2] and [5].

Given sequences of complex numbers $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$, A trigonometric series is any series of the form $a_0 + \sum_1^{\infty}(a_k \cos(kt) + b_k \sin(kt))$, where $t \in R$. Its nth partial sum is of the form $s)n(t) = \sum_{-n}^{n} c_k e^{ikt}$, where if we write $b_0 = 0$, then $c_k = (a_k - ib_k)/2$ and $c_{-k} = (a_k + ib_k)/2$. For this reason, we also write a trigonometric series in the form

$\sum_{-\infty}^{\infty} c_k e^{ikt}$. A trigonometric polynomial on T is an expression of the form $\sum_{-N}^{N} c_k e^{ikt}$ and the largest integer n such that $|c_n|+|c_{-n}|\neq 0$ is called the degree of polynomial. For nonnegative integers k, denote the set of all polynomials of the trigonometric functions e^{ikx} by $h_p^+(\mu)$. For negative integers k, denote the set of all polynomials of the trigonometric functions e^{-ikx} by $h_p^-(\mu)$. Also denote $H_p^+(\mu)$ and $H_p^-(\mu)$ to be the closures of $h_p^+(\mu)$ and $h_p^-(\mu)$ respectively. The following definitions are from [3] and [4].

Definition 1: Let f be a function defined on T. For $1\leq q\leq\infty$, we say that f has a zero of degree q at a point t in T, if $\frac{1}{f}\notin L_I^q$, whenever I is an interval containing t.

Definition 2: Let f be a function defined on T. For $1\leq q\leq\infty$, we say f has a pole of degree q at a point t in T, if $f\notin L_I^q$, whenever I is an interval containing t.

Definition 3: Let f be a function defined on T having a zero of degree q ($1\leq q\leq\infty$)at a point t in T. We say k is the the order of q, if there is a $\delta>0$ and an interval $I=(t-\delta,t+\delta)$ such that $(x-t)^{k-1}\frac{1}{f(x)}\notin L_I^q$, but $(x-t)^k\frac{1}{f(x)}\in L_I^q$.

Definition 4: Let f be a function defined on T. For $1\leq q\leq\infty$, we say f has a zero of degree q of infinite order at a point t in T, if $(x-t)^k\frac{1}{f(x)}\notin L_I^q$, whenever I is an interval containing t.

Definition 5: Let f be a function defined on T. For $1\leq q\leq\infty$, we say f has a pole of degree q of infinite order at a point t in T, if $(x-t)^k f(x)\notin L_I^q$, whenever I is an interval containing t.

Dimension of $H_p^+(\mu)\cap H_p^-(\mu)$ and codimension of $\overline{H_p^+(\mu)\cap H_p^-(\mu)}$

Let μ be a nonnegative measurable function and suppose that for $1\leq p\leq\infty$, $\mu^{\frac{1}{p}}$ has poles of degree p at the points $p_1, p_2, \cdots, p_n$ (for $p=\infty$, we set $\mu^{\frac{1}{p}}=\mu$). Let

$$T(x)=\prod_{j=1}^{n}(e^{ix}-e^{ip_j})^{b_j}, \tag{1}$$

where each b_j $(1 \le j \le n)$ is the order of p at p_j. One can easily see that for every integer $k \ge 0$,

$$e^{ikx}T(x) \in h_p^+(\mu). \qquad (2)$$

Lemma 2.1: Every polynomial in $h_p^+(\mu)$ is a finite linear combination of polynomials of type (2).

Proof: Let $p \in h_p^+(\mu)$. If for some $1 \le j \le n$, t_j is the root of p at p_j, then in term of multiplicity t_j must be greater than or equal to b_j. So the polynomial $(\frac{p}{T})(e^{ix})$ is algebraic, by fundamental theorem of algebra. Also, for every integer $k < 0$,

$$e^{ikx}\overline{T(x)} \in h_p^-(\mu). \qquad (3)$$

Moreover similar to the Lemma 2.1, any polynomial $p \in h_p^-(\mu)$ is a finite linear combination of polynomials of type (3), because $e^{-ix}\overline{T(x)} \in h_p^+(\mu)$.

Suppose that $\mu^{\frac{1}{p}}$ has zeros of degree q at the points $q_1, \cdots q_m$ so that for $1 \le j \le m$ we define a_j to be the order of q at q_j. Put

$$S(x) = \prod_{j=1}^{m}(e^{ix} - e^{iq_j})^{a_j}. \qquad (4)$$

Fix $p \in [1, \infty]$ and define the weight

$$\tilde{\mu}(x) = |T(x)|^p \ \mu(x). \qquad (5)$$

With respect to the notations, we define $\tilde{q}_j$ $(1 \le j \le \tilde{m})$, $\tilde{a}_j$ and

$$\tilde{S}(x) = \prod_{j=1}^{\tilde{m}}(e^{ix} - e^{i\tilde{q}_j})^{\tilde{a}_j}.$$

We also define $A = a_1 + \cdots + a_m$, $\tilde{A} = \tilde{a}_1 + \cdots + \tilde{a}_{\tilde{m}}$ and $B = b_1 + \cdots + b_n$. We have

$$\frac{\overline{T(x)}}{T(x)} = \frac{\prod_{j=1}^{n}(e^{-ix} - e^{-ip_j})^{b_j}}{\prod_{j=1}^{n}(e^{ix} - e^{ip_j})^{b_j}}$$

$$= \frac{\prod_{j=1}^{n} e^{-i(x+p_j)b_j}(e^{ip_j} - e^{ix})_j^b}{\prod_{j=1}^{n}(e^{ix} - e^{ip_j})_j^b} \tag{6}$$

$$= (-1)^B [\prod_{j=1}^{n} e^{-i(b_1 p_1 + \cdots + b_n p_n)}] e^{-iBx}.$$

Now, for any integer r consider the polynomial

$$T_r(x) = \sum_{0}^{A-1} c_j e^{ijx} \tag{7}$$

so that T_r interpolates e^{irx} in the points on which the zeros of $\mu^{\frac{1}{p}}$ occur and the multiplicity of the interpolation at the point q_j is a_j. For any integer r, define

$$g_r(x) = e^{irx} - T_r(x). \tag{8}$$

Then

$$\int_T \frac{|g_r(x)|^q}{|\mu(x)|^q} \mu(x) < \infty,$$

and so

$$\frac{g_r}{\mu} \in L^q(\mu). \tag{9}$$

Moreover if $\int f(x)\overline{g_r(x)}dx = 0$, whenever f is measurable, then $f = 0$ a.e. on T. That is

$$\int f(x)\overline{g_r(x)}dx = 0 \;\Rightarrow\; f = 0 \;\; a.e. \;\; on \;\; T. \tag{10}$$

Also if $\{P_r^+\}$ is a convergent sequence of polynomials in $H_p^+(\mu)$ that is convergent to a function in $H_p^+(\mu)$, then for any non negative integer j,

$$\lim_{r\to\infty}\int_T P_r^+(x)e^{-ijx}dx = \int_T \lim_{r\to\infty} P_r^+(x)e^{ix} g_{-(j+1)}(x)dx. \tag{11}$$

And if if $\{P_r^-\}$ is a convergent sequence of unimodular polynomials in $H_p^-(\mu)$, then for any positive integer j,

$$\lim_{r\to\infty}\int_T P_r^-(x)e^{ijx}dx = \int_T \lim_{r\to\infty} P_r^-(x)\overline{g_{-j}(x)}dx. \tag{12}$$

Note that the convergence mentioned for (11) and (12) are in $L^p(\mu)$-norm. To verify (11), by (8)

$$g_{-j-1}(x) = e^{-i(j+1)x} - T_{-(j+1)}(x),$$

and

$$e^{-ix}\overline{g_{-j-1}(x)} = e^{ijx} - e^{-ix}\overline{T_{-(j+1)}(x)}.$$

Therefore, by (6)

$$\frac{e^{-ix}\overline{g_{-j-1}(x)}}{\mu(x)} \in L^q(\mu), \tag{13}$$

and hence we have (11). Similarly we can proof (12). The following proposition is the summery of our discussion above.

Proposition 2.2: If $\mu^{\frac{1}{p}}$ has at least one pole, then $B > 0$. Hence by (2), if $\{p_n\} \subseteq h_p^+(\mu)$ is a $L^p(\mu)$-convergence sequence of trigonometric polynomials, then $\{p_n T^{-1}\} \subseteq h_p^+(\tilde{\mu})$ is a $L^p(\tilde{\mu})$-convergence sequence of trigonometric polynomials. Also, by (3), if $\{p_n\} \subseteq h_p^-(\mu)$ is a $L^p(\mu)$-convergence sequence of trigonometric polynomials, then $\{p_n T^{-1}\} \subseteq h_p^-(\tilde{\mu})$ is a $L^p(\tilde{\mu})$-convergence sequence of trigonometric polynomials.

Theorem 2.3: if $1 \le p < \infty$, then $H_p^+(\mu) \cap H_p^-(\mu)$ is a finite dimensional linear space. If D is its dimension, then $D = \tilde{A} - B$ whenever $\tilde{A} > B$ and otherwise $D = 0$.

Proof: First we suppose that μ has no pole (meaning $B = 0$). Note that $\{\cdots, e^{-2ix}, e^{-ix}, e^{Aix}, e^{(A+1)ix}, \cdots\}$ is complete in $L^p(\mu)$ and so for every $j \in \{0,1,\cdots,A-1\}$ there is a sequence of polynomial $\{w_{j,k}\}$ converging to e^{ijx} so that if $n \in \{0,1,\cdots,A-1\}$, then

$$\int_T w_{j,k}(x) e^{-inx} dx = 0. \tag{14}$$

Now let m be an arbitrary positive integer and p_m in $H_p^+(\mu) \cap H_p^-(\mu)$ is so that

$$p_m(x) = \sum_{k=-m}^{k=m} c_k e^{ikx}.$$

If $m \ge A-1$, put

$$p_{m_1}(x) = \sum_{k=0}^{A-1} c_k e^{ikx} \quad \tilde{p}_{m_1}(x) = \sum_{k=0}^{A-1} c_k w_{m,k}(x). \tag{15}$$

If $m < A-1$, put

$$p_{m_2}(x) = \sum_{k=0}^{m} c_k e^{ikx} \quad \tilde{p}_{m_2}(x) = \sum_{k=0}^{m} c_k w_{m,k}(x). \tag{16}$$

Since $p_m \in H_p^-(\mu)$, for every nonnegative integer r we have

$$\int_T p_m(x)\overline{g_r(x)}dx = 0.$$

By (11) and (12) if $\{p_n^-\} \subset h_p^-(\mu)$ and $\{p_n^+\} \subset h_p^+(\mu)$ are sequences of polynomials so that in $L^p(\mu)$-norm,

$$\lim_{n\to\infty} p_n^-(x) = \lim_{n\to\infty} p_n^+(x) = p_m(x),$$

then for every positive integer r,

$$\lim_{n\to\infty} \int_T p_n^-(x)e^{irx}dx = c_r,$$

and for every integer $r \le 0$,

$$\lim_{n\to\infty} \int_T p_n^+(x)e^{-irx}dx = \overline{c_r}.$$

Next, suppose that $B > 0$. For $\tilde{\mu}$ be as in (5), define the set $H_p^+(\tilde{\mu}).F$ to be the set of all functions $f.F$ so that $f \in H_p^+(\tilde{\mu})$. We similarly define $H_p^-(\tilde{\mu}).F$ the set of all functions $f.F$ so that $f \in H_p^-(\tilde{\mu})$. By Proposition 1, $H_p^+(\mu) = H_p^+(\tilde{\mu}).T$ and $H_p^-(\mu) = H_p^-(\tilde{\mu}).\overline{T}$. Therefore the sets $H_p^+(\mu) \cap H_p^-(\mu)$ and $(H_p^+(\tilde{\mu}).e^{iBx}) \cap H_p^-(\tilde{\mu})$ are equal, by (6).

If $\tilde{A} \le B$, then the system $\{e^{kix}\}_{k<0} \cup \{e^{kix}\}_{k\ge B}$ is minimal in the space $L^p(\tilde{\mu})$. So we may assume that $\tilde{A} > B$. Now similar to the above argument, if for arbitrary positive integer m we let q_m be a in $(H_p^+(\tilde{\mu}).e^{iBx}) \cap H_p^-(\tilde{\mu})$, then all the calculations for p_{m_1} and p_{m_2} defined in (15) and (16) are valid for $\tilde{p}_{m_1}$ and $\tilde{p}_{m_2}$. Therefore we have the result.

Next, we present the value for the Codimension of $\overline{H_p^+(\mu) \cup H_P^-(\mu)}$, where μ is the nonnegative measurable function as in Theorem 1.

Theorem 2.4: If D_1 is the Codimension of $\overline{H_p^+(\mu)+H_P^-(\mu)}$, the $D_1 = B-\tilde{A}$ whenever $\tilde{A} < B$ and otherwise $D_1 = 0$.

Proof: first note that if $p \in h_p^-(\mu)$, then by the Lemma 1 and (6), p is a finite linear combination of polynomials $e^{ikx}.T(x) \in h_p^-(\mu)$, whenever $k \le -B-1$. Hence D' must be equal to the codimension of $\overline{H_p^+(\tilde{\mu})+H_P^-(\tilde{\mu}).e^{-iBx}}$.

Theorem 2.5: Let $\tilde{A} > B$ and let $\tilde{h}_p^-(\mu)$ be the linear subspace in $L^p(\mu)$ of polynomials of the trigonometric functions e^{ikx} $(k < B-\tilde{A})$. Let $\tilde{H}_p^-(\mu)$ be the closure of $\tilde{h}_p^-(\mu)$. Then the dimension of $\tilde{H}_p^-(\mu) \cap H_p^-(\mu)$ must be zero.

Proof: By the Lemma 1, the systems $\{e^{ikx}T(x)\}_{k=0}^{\infty}$ and $\{e^{-ikx}\overline{T(x)}\}_{k=\tilde{A}-B+1}^{\infty}$ are complete in $H_p^+(\mu)$ and $\tilde{H}_p^-(\mu)$ respectively. Therefore, by (6), the system $\{e^{-ikx}T(x)\}_{k=\tilde{A}+1}^{\infty}$ must be complete in $\tilde{H}_p^-(\mu)$. Thus the systems $\{e^{ikx}\}_{k=0}^{\infty}$ and $\{e^{-ikx}\}_{k=\tilde{A}+1}^{\infty}$ must respectively be complete in $H_p^+(\tilde{\mu})$ and $\tilde{H}_p^+(\tilde{\mu})$. Hence the system $\{e^{ikx}\}_{k=0}^{\infty} \cup \{e^{-ikx}\}_{k=\tilde{A}+1}^{\infty}$ is complete minimal in $L^p(\tilde{\mu})$.

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