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On Differential Sandwich Theorems for p-valent Functions Defined by Certain Fractional Derivative Operator

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Abstract

The purpose of this paper is to obtain some applications of first order differential subordination and super ordination results involving certain fractional derivative operator for p-valent functions in the open unit disk.

Keywords: p-valent function, Differential subordination, Differential superordination, fractional derivative operators.

Mathematics subject classification: 30C45, 26A33

Introduction and Preliminaries

Let $\mathcal{H}(\mathcal{U})$ denote the class of analytic functions in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and let $\mathcal{H}[a, p]$ denote the subclass of the functions $f \in \mathcal{H}(\mathcal{U})$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots \quad (a \in \mathbb{C}, p \in N)$$

Also, let A(p) be the class of functions $f \in \mathcal{H}(\mathcal{U})$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \ , p \in N \tag{1.1}$$

and set $A \equiv A(1)$.

Let $f, g \in \mathcal{H}(\mathcal{U})$, we say that the function f is subordinate to g, if there exist a Schwarz function w, analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathcal{U})$, such that $f(z) = g(w(z))$ for all $z \in \mathcal{U}$.

This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$. It is well known that, if the function g is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let $p(z), h(z) \in \mathcal{H}(\mathcal{U})$, and let $\Phi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. If $p(z)$ and $\Phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent functions, and if $p(z)$ satisfies the second-order superordination

$$h(z) \prec \Phi(p(z), zp'(z), z^2 p''(z); z) \tag{1.2}$$

then $p(z)$ is called to be a solution of the differential superordination (1.2). (If f(z) is subordinatnate to $g(z)$, then $g(z)$ is called to be superordinate to f(z)). An analytic function $\tilde{q}(z)$ is called a subordinant if $q(z) \prec p(z)$ for all p(z) satisfies (1.2) . An univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.2) is said to be the best subordinant.

Recently, Miller and Mocanu [5] obtained conditions on $h(z), q(z)$ and Φ for which the following implication holds true:

$$h(z) \prec \Phi(p(z), zp'(z), z^2 p''(z); z) \Longrightarrow q(z) \prec p(z)$$

with the results of Miller and Mocanu [5], Bulboaca [2] investigated certain classes of first order differential superordinations as well as superordination-preserving integral operators [3]. Ali et al. [1] used the results obtained by Bulboaca [3] and gave the sufficient conditions for certain normalized analytic functions f(z) to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. [8] obtained sufficient conditions for a normalized analytic functions to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{(f(z))^2} \prec q_2(z)$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

Let ${}_2F_1(a, b; c; z)$ be the Gauss hypergeometric function defined for $z \in \mathcal{U}$ by, (see Srivastava and Karlsson [9])

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} \tag{1.3}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0 \\ \lambda(\lambda+1)(\lambda+2) \dots (\lambda+n-1), & n \in \mathbb{N} \end{cases} \tag{1.4}$$

for $\lambda \neq 0, -1, -2, \ldots$

We recall the following definitions of fractional derivative operators which were used by Owa [6], (see also [7]) as follows:

Definition 1.1: The fractional derivative operator of order λ is defined,

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi \tag{1.5}$$

where $0 \leq \lambda < 1$, $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 1.2: Let $0 \leq \lambda < 1$, and $\mu, \eta \in \mathbb{R}$. Then, in terms of the familiar Gauss's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz} \left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi)\, {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{\xi}{z} \right) d\xi \right) \tag{1.6}$$

where $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin with the order $f(z) = O(|z|^{\varepsilon})$, $z \to 0$, where $\varepsilon > \max\{0, \mu-\eta\} - 1$, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 1.3: Under the hypotheses of Definition 1.2, the fractional derivative operator $J_{0,Z}^{\lambda+m,\mu+m,\eta+m} f(z)$ of a function $f(z)$ is defined by

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z) \tag{1.7}$$

Notice that

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^{\lambda} f(z), 0 \leq \lambda < 1 \tag{1.8}$$

With the aid of the above definitions, we define a modification of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ by

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z) \tag{1.9}$$

for $f(z) \in A(p)$ and $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda, \mu) - p - 1$; $p \in N$. Then it is observed that $M_{0,z}^{\lambda,\mu,\eta} f(z)$ maps $A(p)$ onto itself as follows:

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda, \mu, \eta, p)\, a_{p+n} z^{p+n} \tag{1.10}$$

where

$$\delta_n(\lambda,\mu,\eta,p) = \frac{(p+1)_n(p+1-\mu+\eta)_n}{(p+1-\mu)_n(p+1-\lambda+\eta)_n} \quad (1.11)$$

It is easily verified from (1.10) that

$$z\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)' = (p-\mu)\,M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z) + \mu\,M_{0,z}^{\lambda,\mu,\eta}f(z) \quad (1.12)$$

Notice that

$$M_{0,z}^{0,0,\eta}f(z) = f(z),$$

and

$$M_{0,z}^{1,1,\eta}f(z) = \frac{zf'(z)}{p}$$

The object of this paper is to derive several subordination results, superordination results and sandwich results for p-valent functions involving certain fractional derivative operator. Some special cases are also considered.

In order to prove our results we mention to the following known results which shall be used in the sequel.

Lemma 1.4 [7]: Let $\lambda,\mu,\eta \in \mathbb{R}$, such that $\lambda \geq 0$ and $k > \max\{0,\mu-\eta\}-1$. Then

$$J_{0,z}^{\lambda,\mu,\eta}z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)}z^{k-\mu} \quad (1.13)$$

Definition 1.5 [5]: Denoted by Q the set of all functions f that are analytic and injective in $\bar{\mathcal{U}} - E(f)$ where

$$E(f) = \{\xi \in \partial\mathcal{U} : \lim_{z\to\infty} f(z) = \infty\}$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial\mathcal{U} - E(f)$.

Lemma 1.6 [4]: Let the function q be univalent in the open unit disk $\mathcal{U}$, and θ and φ be analytic in a domain D containing $q(\mathcal{U})$ with $\varphi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that Q is starlike univalent in $\mathcal{U}$, and

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0 \text{ for } z \in \mathcal{U}$$

If

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z))$$

Then $p(z) \prec q(z)$ and q is the best dominant.

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 1.6, Shanmugam et al. [8] obtained the following lemma.

Lemma 1.7 [8]: Let q be univalent in the open unit disk $\mathcal{U}$ with $q(0) = 1$ and $\alpha,\gamma \in \mathbb{C}$.

Further assume that

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\operatorname{Re}\left(\frac{\alpha}{\gamma}\right)\right\}$$

If $p(z)$ is analytic in $\mathcal{U}$, and

$$\alpha p(z) + \gamma z p'(z) \prec \alpha q(z) + \gamma z q'(z)$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.8 [2]: Let the function q be univalent in the open unit disk $\mathcal{U}$, and θ and φ be analytic in a domain D containing $q(\mathcal{U})$ with $\varphi(w) \neq 0$ when $w \in q(\mathcal{U})$. Suppose that

$$\operatorname{Re}\left(\frac{\theta'(q(z))}{\varphi(q(z))}\right) > 0 \text{ for } z \in \mathcal{U}$$

$zq'(z)\varphi\big(q(z)\big)$ is starlike univalent in $\mathcal{U}$.

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$ with $p(\mathcal{U}) \subseteq D$, and $\theta\big(p(z)\big) + zp'(z)\varphi(p(z))$ is univalent in $\mathcal{U}$, and

$$\theta\big(q(z)\big) + zq'(z)\varphi(q(z)) \prec \theta\big(p(z)\big) + zp'(z)\varphi(p(z))$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 1.8, Shanmugam et al. [8] obtained the following lemma.

Lemma 1.9 [8]: Let q be univalent in the open unit disk $\mathcal{U}$ with $q(0) = 1$. Let $\alpha, \gamma \in \mathbb{C}$ and $\operatorname{Re}\left(\frac{\alpha}{\gamma}\right) > 0$. If $p(z) \in \mathcal{H}[q(0),1] \cap Q, \alpha p(z) + \gamma z p'(z)$ is univalent in $\mathcal{U}$, and

$$\alpha q(z) + \gamma z q'(z) \prec \alpha p(z) + \gamma z p'(z)$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Subordination and superordination for p-valent functions

We begin with the following result involving differential subordination between analytic functions.

Theorem 2.1: Let q be univalent in $\mathcal{U}$ with $q(0) = 1$, and suppose that

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\operatorname{Re}\left(\frac{1}{\gamma}\right)\right\} \tag{2.1}$$

If $f(z) \in A(p)$, and

$$F_{\lambda,\mu,\eta}(\gamma, f)(z) = [1+\gamma(p-\mu+1)]\frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} +$$

$$\gamma(p-\mu-1)\frac{z^p\,M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}-2\gamma(p-\mu)\frac{z^p\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)^2}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^3} \tag{2.2}$$

If q satisfies the following subordination:

$$F_{\lambda,\mu,\eta}(\gamma,f)(z)\prec q(z)+\gamma zq'(z) \tag{2.3}$$

$$(\lambda\geq 0;\ \mu<p+1;\ \eta>\max(\lambda,\mu)-p-1;\ p\in\mathbb{N};\ \gamma\in\mathbb{C})$$

then

$$\frac{z^p\,M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}\prec q(z) \tag{2.4}$$

and q is the best dominant.

Proof: Let the function $p(z)$ be defined by

$$p(z)=\frac{z^p\,M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}$$

So that, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)}=p+\frac{z\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-\frac{2z\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta}f(z)} \tag{2.5}$$

By using the identity (1.12) a simple computation shows that

$$[1+\gamma(p-\mu+1)]\frac{z^p\,M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}+\gamma(p-\mu-1)\frac{z^p\,M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^2}-$$

$$2\gamma(p-\mu)\frac{z^p\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)^2}{\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)^3}=p(z)+\gamma zp'(z)$$

The assertion (2.4) of Theorem 2.1 now follows by an application of Lemma 1.7, with $\alpha=1$.

Remark 1: For the choice $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, in Theorem 2.1, we get the following Corollary.

Corollary 2.2: Let $-1\leq B<A\leq 1$, and suppose that

$$\operatorname{Re}\left(\frac{1-Bz}{1+Bz}\right)>\max\left\{0,-\operatorname{Re}\left(\frac{1}{\gamma}\right)\right\} \tag{2.6}$$

If $f(z)\in A(p)$, and

$$F_{\lambda,\mu,\eta}(\gamma, f)(z) \prec \frac{1+Az}{1+Bz} + \frac{\gamma(A-B)z}{(1+Bz)^2}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in N;\ \gamma \in \mathbb{C})$$

where $F_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (2.2), then

$$\frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Next, by appealing to Lemma 1.9 of the preceding section, we prove the following.

Theorem 2.3: Let q be convex in $\mathcal{U}$ and $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$. If $f(z) \in A(p)$,

$$0 \neq \frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} \in \mathcal{H}[1,1] \cap Q$$

and $F_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q(z) + \gamma z q'(z) \prec F_{\lambda,\mu,\eta}(\gamma, f)(z) \tag{2.7}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in N;\ \gamma \in \mathbb{C})$$

implies

$$q(z) \prec \frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} \tag{2.8}$$

and q is the best subordinant where $F_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (2.2).

Proof: Let the function $p(z)$ be defined by

$$p(z) = \frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2}$$

Then from the assumption of Theorem 2.3, the function $p(z)$ is analytic in $\mathcal{U}$ and (2.5) holds. Hence, the subordination (2.7) is equivalent to

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z)$$

The assertion (2.8) of Theorem 2.3 now follows by an application of Lemma 1.9.

Combining Theorem 2.1 and Theorem 2.3, we get the following sandwich theorem.

Theorem 2.4: Let q_1 and q_2 be convex functions in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$. If $f(z) \in A(p)$ such that

$$\frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} \in \mathcal{H}[1,1] \cap Q$$

and $F_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec F_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in N;\ \gamma \in \mathbb{C})$$

implies

$$q_1(z) \prec \frac{z^p\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} \prec q_2(z) \tag{2.9}$$

and q_1 and q_2 are respectively the best subordinant and the best dominant where $F_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (2.2).

Remark 2: For $\lambda = \mu = 0$ in Theorem 2.4, we get the following result.

Corollary 2.5: Let q_1 and q_2 be convex functions in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$. If $f(z) \in A(p)$ such that

$$\frac{z^{p+1} f\prime(z)}{p\left(f(z)\right)^2} \in \mathcal{H}[1,1] \cap Q$$

and let

$$F_1(\gamma, f)(z) = [1 + \gamma(p+1)]\frac{z^{p+1} f\prime(z)}{p\left(f(z)\right)^2} + \gamma\frac{z^{p+2} f\prime\prime(z)}{p\left(f(z)\right)^2} - 2\gamma\frac{z^{p+2}\left(f\prime(z)\right)^2}{p\left(f(z)\right)^3}, p \in N$$

is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec F_1(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)$$

implies

$$q_1(z) \prec \frac{z^{p+1}\, f\prime(z)}{p\left(f(z)\right)^2} \prec q_2(z) \tag{2.10}$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Theorem 2.6: Let q be univalent in $\mathcal{U}$ with $q(0) = 1$, and assume that (2.1) holds. If $f(z) \in A(p)$, and

$$G_{\lambda,\mu,\eta}(\gamma, f)(z) = (1+\gamma)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} + \gamma(p - \mu - 1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} -$$

$$\gamma(p-\mu)\frac{\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)\right)^2}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} \tag{2.11}$$

If q satisfies the following subordination:

$$G_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q(z) + \gamma z q'(z)$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in N;\ \gamma \in \mathbb{C})$$

then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec q(z) \tag{2.12}$$

and q is the best dominant.

Proof: Let the function p(z) be defined by

$$p(z) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}$$

So that, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)} = \frac{z\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)\right)'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} - \frac{z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \tag{2.13}$$

By using the identity (1.12) a simple computation shows that

$$(1+\gamma)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} + \gamma(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} -$$

$$\gamma(p-\mu)\frac{\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)\right)^2}{\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)^2} = p(z) + \gamma z p'(z) \tag{2.14}$$

The assertion (2.12) of Theorem 2.6 now follows by an application of Lemma 1.7, with $\alpha = 1$.

Remark 3: For the choice $q(z) = \frac{1+Az}{1+Bz}$ $, -1 \leq B < A \leq 1$, in Theorem 2.6, we get the following result.

Corollary 2.7: Let $-1 \leq B < A \leq 1$, and assume that (2.6) holds. If $f(z) \in A(p)$, and

$$G_{\lambda,\mu,\eta}(\gamma, f)(z) \prec \frac{1+Az}{1+Bz} + \frac{\gamma(A-B)z}{(1+Bz)^2}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in N;\ \gamma \in \mathbb{C})$$

where $G_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (2.11), then

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Next, by appealing to Lemma 1.9 of the preceding section, we prove the following.

Theorem 2.8: Let q be convex in $\mathcal{U}$ and $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$. If $f(z) \in A(p)$,

$$0 \neq \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \in \mathcal{H}[1,1] \cap Q$$

and $G_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q(z) + \gamma z q'(z) \prec G_{\lambda,\mu,\eta}(\gamma, f)(z) \tag{2.15}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in N;\ \gamma \in \mathbb{C})$$

implies

$$q(z) \prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \tag{2.16}$$

and q is the best subordinant where $G_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (2.11).

Proof: Let the function $p(z)$ be defined by

$$p(z) = \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}$$

Then from the assumption of Theorem 2.8, the function $p(z)$ is analytic in $\mathcal{U}$ and (2.13) holds. Hence, the subordination (2.15) is equivalent to

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z)$$

The assertion (2.16) of Theorem 2.8 now follows by an application of Lemma 1.9.

Combining Theorem 2.6 and Theorem 2.8, we get the following sandwich theorem.

Theorem 2.9: Let q_1 and q_2 be convex functions in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$. If $f(z) \in A(p)$ such that

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \in \mathcal{H}[1,1] \cap Q$$

and $G_{\lambda,\mu,\eta}(\gamma, f)(z)$ is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec G_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)$$

$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in N;\ \gamma \in \mathbb{C})$

implies

$$q_1(z) \prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant where $G_{\lambda,\mu,\eta}(\gamma, f)(z)$ is as defined in (2.11).

Remark 4: For $\lambda = \mu = 0$ in Theorem 2.9, we get the following result.

Theorem 2.10: Let q_1 and q_2 be convex functions in $\mathcal{U}$ with $q_1(0) = q_2(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$. If $f(z) \in A(p)$ such that

$$\frac{zf'(z)}{pf(z)} \in \mathcal{H}[1,1] \cap Q$$

and let

$$G_1(\gamma, f)(z) = \frac{(1+\gamma)}{p}\frac{zf'(z)}{f(z)} + \frac{\gamma}{p}\left\{\frac{z^2 f''(z)}{f(z)} - \frac{z^2\left(f'(z)\right)^2}{\left(f(z)\right)^2}\right\}, p \in N$$

is univalent in $\mathcal{U}$, then

$$q_1(z) + \gamma z q_1'(z) \prec G_1(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)$$

implies

$$q_1(z) \prec \frac{zf'(z)}{pf(z)} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

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