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On the Model of the Price of an Option base on Stochastic Volatility

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Abstract

The price of an option's model with a stochastic volatility instead of the Black-Schole's model which uses a constant volatility, that is $\sigma_{S_i(t)} = \lim_{T\to 0} v(t,T,1) = v(t,0,1)$ is considered herein. The value of an option based on such model is obtained by solving an elliptic partial differential equation. Prices of an option based on some assumptions are obtained.

Keywords: stochastic volatility, option pricing, option, elliptic partial differential equation.

Introduction

Option pricing models with structure volatility are more realistic than the Black-Schole's model which uses a constant volatility. The prices of option based on such models can be obtained by solving a parabolic partial differential equation (Osu and Okoroafor, 2007) The option price obtained using Black- Schole's model(Black and Schole, 1973) are not consistent with observed option price. Ledoit, et al (2002) have shown that the Black-Schole's implied volatilities of at- the- money option converge to the underlying assets instantaneous (Stochastic) volatility as the time to maturity goes to zero. That is, it is just a convenient and well known mapping from prices of option (that are actually traded) to volatilities (that are typically quoted). The mapping particularly, is continuous in the strike price and time to maturity, and in such that volatility of the strike price is equal to the limit as time to maturity goes to zero of at-the-money volatility.

The traditional approach to pricing options on stocks with stochastic volatility starts by specifying the point process for the stock price and its volatility and making some assumption about the market price of volatility risk. Stein and Stein (1991), Heston (1993), Hull and White(1987), among others had followed this approach which uses the risk adjusted point process followed by the stock price and its volatility to price options in closed form if possible, but more likely with numeric methods. This model are typically calibrated to the price of a few option or estimated from the time series of stock prices.

Derman and Kani (1997) offered a model, the dynamics of what they term the " Local Volatility Surface" and found a no-arbitrage condition that it must satisfy. Their model assumed that the stock price volatility is a deterministic function of the stock price itself and time, so that the stock price is the only source of uncertainty. This assumption is in the same spirit with Osu, (2010) in the consideration of a stochastic model of the variation of the Capital Market Price, whereby precise conditions are obtained which determine the equilibrium price.

Local volatilities $\mathfrak{z}(t,s,k)$ are defined formally (Ledoit, 2002) at time t as the stock volatility at future times and stock level $S(s) = K$ that would price all observed options correctly by

$$\mathfrak{z}(t,s,k)^2 = \frac{2\frac{\partial c}{\partial s}(t,s,k) + (r-q)k\frac{\partial c}{\partial l} + qC(t,s,k)}{\mathrm{k}^2\frac{\partial^2}{\partial \mathrm{k}^2}(t,s,k)}, \quad (1)$$

where $C(t,s,k)$ denotes the time t price of a call option with maturity at time s and strike price K.

The problem with Derman and Kani (1997) approach is that local volatilities are not readily measurable and that there is no explicit relationship between option and local volatilities. The resulting arbitrage free prices and local volatility are rather complicated.

Ledoit et, al, (2002) offered a new approach for pricing options on assets with stochastic volatility. Their approach simply requires as inputs the stock price and local volatility the exotic option is to be priced, as well as estimates of the volatilities of the implied volatilities.

On the other hand Ikonen and Toivanen (2004) studied the operator splitting methods for performing time stepping after a finite difference space discretization is done for pricing America options using Heston's Model. With stochastic volatility models it makes efficient solution procedures much simpler if the early exercise constraint can be treated in a separate paper.

In this paper, we consider the model presented by Heston,(1993) and obtain a solution of an elliptic partial differential, which variables are time, the underlying asset value, and the volatility. We also obtain prices of options based on this model.

Model Formulation

The choice of option sells, in-active, or sell, is represented as $S_i(t) \in \{-1,0,1\}$. Each decision, $S_i(t) \in \{-1,0,1\}$, is determined by option's strategy. The value in the interval between $(-1,0,1)$ is normalized such that it can be seen as probability, i.e

$$P[S_i(t) = s] = \frac{\sum_{\mathrm{i}}^{\mathrm{S_i(t)=s}} \mathrm{X_i(t)}}{\sum_i X_i}, \quad (2)$$

with total possibility follows:

$$\sum P[S_i(t) = s] = 1. \quad (3)$$

Where on the variable of influence strength X_i, each option affects and is affected by its surroundings.

Given a finite time horizon $T > 0$, we consider herein a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a standard Wiener process $W = (\{W_i(t), \ldots, W_n(t)\}, 0 \leq t \leq T)$ valued in R^n, and generating the (P-augmented) filtration $\mathcal{F}$. The financial market consists of non-risky asset $S(0)$ normalized to unity, i.e. $S(0) = 1$, and n risky assets with price process $S = (S_i(t), \ldots, S_n(t))$ whose dynamics is defined by a stochastic differential equation (Etheridge, 2002)

$$dS(t) = \mu S(t) + \sigma S(t) dW(t), \quad (4)$$

with solution via Ito's formula given as;

$$S(t) = S(0) exp\left\{\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right) t\right\}, \forall\, t\epsilon\, [0, 1] \quad (5)$$

Suppose the price of a certain option is transformed into a price at a time t, let K be the initial price, and $S(t)$ be the prince of option at time t.

Then $k - S(t)$ $(for\ k > S)$ is the contingent claim after a given time horizon t. Suppose further that the change in price is "autocatalytic" (that is the change is due to the option value itself at time t and not necessarily option price), then $\frac{dS}{dt}$ is proportional both to $S(t)$ and $k - S(t)$ and we have;

$$\frac{dS(t)}{dt} = \mu S(t)(K - S(t)), \quad (6)$$

where μ is a positive constant (the drift in economic parlance). Assume in addition that the stock price change is characterized by white noise, M_t (i.e. uncertainty, which is usually the case), that is $d\ W_t \sim N\ (0, \Delta_t)$, then $M_t dt = \mathfrak{z}_t dW(t)$ so we can write (4) as;

$$dS(t) = \mu S(t)\big(K - S(t)\big)dt + \mathfrak{z}_t S(t)\big(K - S(t)\big)dW_1, \quad (7)$$

$$d\mathfrak{z}_t = \alpha(\beta - \mathfrak{z}_t^2)dt + \gamma \mathfrak{z}_t dW_2. \quad (8)$$

The set of equations above are stochastic differential equations similar to that in Heston's model. They define the stock price process S_t and the variance process $\mathfrak{z}_t$. Equation (7) models the stock price process S_t. The parameter μ is the average rate of the deterministic growth of the stock price and $\mathfrak{z}_t$ is the standard deviation (volatility) of the stock returns $\frac{dS}{S(K-S)}$. The model for the variance process $\mathfrak{z}_t$ is given by (8). The volatility of the variance process $\mathfrak{z}_t$ is denoted by γ (volatility of volatility) and the variance will drift back to mean value $\beta > 0$ at a rate $\alpha > o$. These two processes contain randomness, that is, W_1 and W_2 are Wiener processes and furthermore, the processes are linked together with the correlation factor $\rho \in [-1,1]$ (Clarke and Parrott, 1999).

For the price of the America option, a two-dimensional parabolic partial differential equation can be derived using the previous stochastic volatility model (Zvan et al, 1998), thus;

$$-\frac{\partial u}{\partial t} = \frac{\xi^2}{2}\left(s(k-s)\right)^2\frac{\partial^2 u}{\partial s^2} + \rho\gamma\xi^2\left(s(k-s)\right)^2\frac{\partial^2 u}{\partial s\partial \xi} + \frac{\gamma^2}{2}\xi^2\frac{\partial^2 u}{\partial \xi^2} + rs(k-s)\frac{\partial u}{\partial s} - \{\alpha(\beta-\xi^2)\varphi\gamma\xi\}\frac{\partial u}{\partial \xi} - ru, \tag{9}$$

where the parameter $\varphi = 0$ is the so called market price of risk. The original option pricing problem is a final value problem, since the option is known at expiration.

Lemma 1

The value of the option at time t under the volatility with interest rate$r$ given equation (9) is :

$$u(s,\xi) = \sum_{n=1}^{\infty}\left\{\begin{matrix}[C_n cos\left(\frac{n\pi\xi}{k}\right) + \\ D_n sin\left(\frac{n\pi\xi}{k}\right)]\,cos\left(\frac{n\pi s}{k}\right)\end{matrix}\right\} e^{-rs(k-s)\vartheta - \alpha(\beta-\xi^2)\eta} \tag{10}$$

Proof

With equation (9) is associated a characteristic equation;

$$s(k-s)d\xi^2 - 2\rho\gamma ds d\xi + (s(k-s))^{-1}\gamma^2 ds^2 = 0, \tag{11}$$

which can be factored and written in the form

$$(d\xi - \lambda_1\, ds)(d\xi - \lambda_2 ds) = 0. \tag{12}$$

λ_1, λ_2 are the roots of

$$s(k-s)\lambda^2 - 2\rho\gamma\lambda + (s(k-s))^{-1}\gamma^2 = 0. \tag{13}$$

The solutions to (13) will be certain linear transformations

$$\xi - \lambda_1 s = \vartheta + \eta,\ \xi - \lambda_2 s = \vartheta - \eta, \tag{14}$$

which transform (9) into the elliptic form

$$u_{\vartheta\vartheta} + u_{\eta\eta} + rs(k-s)u_{\vartheta} + \{\alpha(\beta-\xi^2)\}u_{\eta} + Ru = -u_t. \tag{15}$$

We now introduce a new dependent function v defined by

$$u = ve^{-rs(k-s)\vartheta - \alpha(\beta-\xi^2)\eta}, \tag{16}$$

such that (15) becomes for $R = 0$ (John, 1974) and for $u_t = 0$(since t is suppressed)

$$v_{ss} + v_{\xi\xi} = 0, \tag{17}$$

with initial value

$$v(s, \xi, 0) = \max(E - s, 0), \tag{18}$$

where E is the exercise price, and boundary conditions are described as

$$v(0, \xi, t) = E, (\xi, t) \in [0, \zeta] \times [0, T], \tag{19}$$

$$v(s, 0, t) = \max(E - s, 0), (s, t) \in [0, S] \times [0, T], \tag{20}$$

with $\lim_{s \to E} v(k, 0, t) = v(s, 0, t)$, for $E = k$

$$\frac{\partial v(S, \xi, t)}{\partial s} = 0, (\xi, t) \in [0, \zeta] \times [0, T], \tag{21}$$

$$\frac{\partial v(s, \zeta, t)}{\partial \xi} = 0, (s, t) \in [0, S] \times [0, T]. \tag{22}$$

We now apply the method of separation of variables. Namely we suppose the solution will be of the form $v = h(s)g(\xi)$. Substituting into (17), we are led to two ordinary differential equations for h and g as;

$$h'' + \delta^2 h = 0, \tag{23}$$

$$g'' + \delta^2 g = 0. \tag{24}$$

The general solution of (23) is $h(s) = A\cos\delta s + \mathrm{B}\sin\delta s$, where A and B are constants; the condition $\frac{\partial v(S,\xi,t)}{\partial s} = 0$, implies that $h'(0) = 0$, so that $B = 0$. Similarly, the condition $v(k, 0, t) = 0$ shows that $h(k) = 0$ and hence, unless $A = 0$(this would give $v = 0$), $\cos k\delta = 0$.

The only possible values of δ are $\frac{n\pi}{k}$, where n is a range over all positive values since we have no method of rejecting any values of n. The equation is linear and hence the sum of constant multiples of solutions is also a solution. Therefore, possible form of v given the solution of (24) is

$$v = \sum_{n=1}^{\infty} \left\{ [C_n cos\left(\frac{n\pi\xi}{k}\right) + D_n sin\left(\frac{n\pi\xi}{k}\right)] \, cos\left(\frac{n\pi s}{k}\right) \right\}. \tag{25}$$

Combining (25) with (16) gives (10) as required. Figure 1 below describes the behavior of the kernel u for $s \geq 0$.

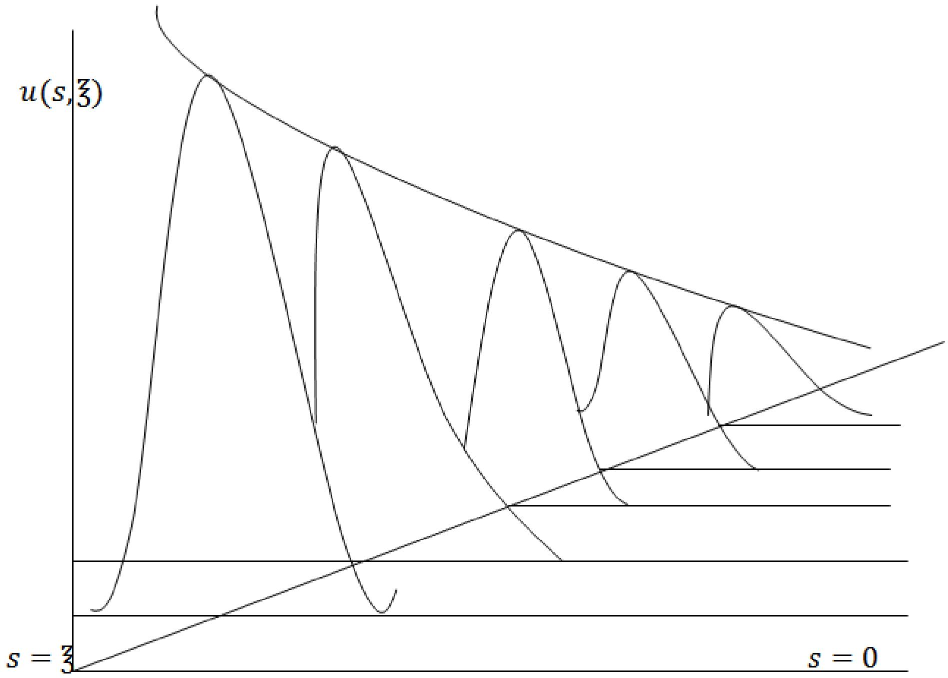

Figure 1: The behaviour of an option value u with different value the price s and variance ξ

We now attempt a solution of the stochastic differential equation (7) to obtain the price s of an option.

Proposition 1: Given the SDE in equation (7) the price s is given by;

$$s(t) = \frac{kS_0 e^{\left\{k\left(\xi_t W_1 + \left(\mu - \frac{\xi_t^2}{2}\right)t\right)\right\}}}{1 + k e^{\left\{k\left(\xi_t W_1 + \left(\mu - \frac{\xi_t^2}{2}\right)t\right)\right\}}} = \frac{kS_0}{e^{-\left\{k\left(\xi_t W_1 + \left(\mu - \frac{\xi_t^2}{2}\right)t\right)\right\}} + k}. \quad (26)$$

Proof : It is easy to see using Ito's formula that (7) becomes

$$\ln\left(\frac{s}{k-s}\right) = k\left(\xi_t W_1 + \left(\mu - \frac{\xi_t^2}{2}\right)t\right), \quad (27)$$

hence (26) follows as required. Figure 2 below illustrates the limit behaviours of equation (26) with time t in seconds.

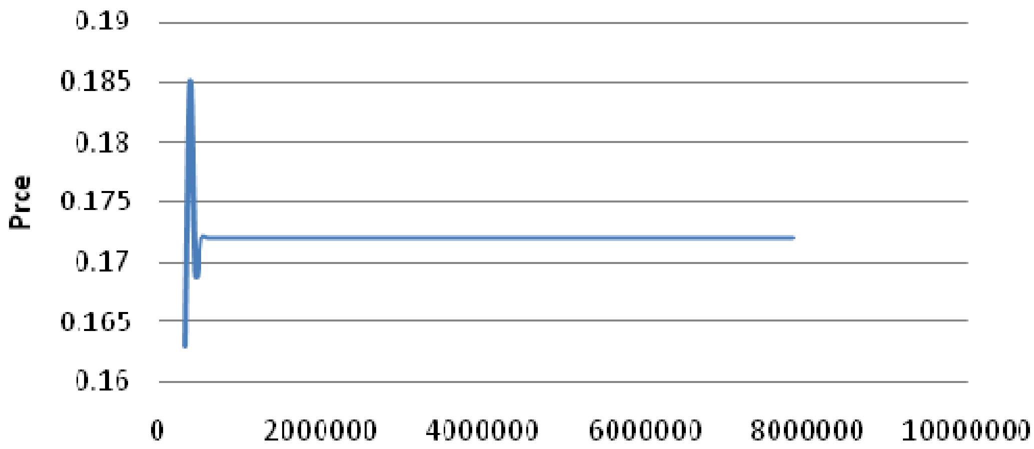

Figure 2: The case of damped fluctuation as a convergent price path is produced.

Discussion and conclusion

We have considered Heston model in this paper to obtain the value of an option given the price s. Notice that as $s \to 0$ and $\xi \to 0$, $u \to \sum_{n=1}^{\infty} C_n e^{-\alpha\beta\vartheta}$, that is the value of an option u, tends to a constant (probably to k-the strike price) depending on the mean value β at a rate α as the price s at time $t = 0$ tends to 0. Again as $s \to \infty$, $u \to 0$; that is as the price s increases with time $t > 0$, the value of the option dampens. Each successive cycle has a smaller amplitude than the preceding one (see figure 1), much as the way a ripple dies down. Also for $s = k$ and $\beta = \xi$, $u = v$ (as in 25). On the other hand, as $t \to 0$, and $W \to 0$, $s \to \frac{ks_0}{1+k}$. Also as $t \to \infty$, $s(t) \to s_0$, as seen in figure 2.

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