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On Products of Conjugate Secondary Range k-Hermitian Matrices

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Abstract

The concept of products of conjugate secondary range k-hermitian matrix [4] (con-s-k-EP) is introduced. We explore the conditions for the product of con-s-k-EP_r matrices to be con-s-k-EP_r.

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Introduction

Let C_{nxn} be the space of nxn complex matrices of order n. Let C_n be the space of all complex n-tuples. For $A \in C_{nxn}$, let $\bar{A}$, A^T, A^*, A^S, $A^\dagger$, R(A), N(A) and ρ (A) denote the conjugate, transpose, conjugate transpose, secondary transpose, conjugate secondary transpose, Moore-Penrose inverse, range spaces, null spaces and rank of A respectively. A solution X of the equation AXA = A is denoted by A^- (Generalized Inverses of A). For $A \in C_{nxn}$, the Moore Penrose inverse $A^\dagger$ of A is the unique solution of the equations

$$AXA = A, XAX = X, [AX]^* = AX, [XA]^* = XA \ [2].$$

Anna Lee [1] has initiated the study of secondary symmetric matrices that is matrices whose entries are symmetric about the secondary diagonal. Cantoni Antono and Butler Paul [3] have studied per-symmetric matrices that is matrices are symmetric about both the diagonals and their applications to communication theory. In [1] Anna Lee has shown that for a complex matrix A, the usual transpose A^T and A^S are related as $A^S = VA^TV$ where 'V' is the permutation matrix with units in its secondary diagonal. Also the conjugate transpose A^* and the secondary conjugate transpose A^{-S} are related as $A^{-S} = VA^*V$. Throughout let 'k' be fixed product of disjoint transpositions in $S_n=\{1,2,3\ldots n\}$ and 'K' be the associated permutation matrix.

Products of conjugate secondary range k-hermitian matrix

It is well known that the product of non singular matrix is non singular. In general, the product of symmetric, hermitian, normal, EP, con-EP con-k-EP and con-s-EP matrices. Similarly, the product of con-s-k-EP matrices need not be con-s-k-EP. For instance let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

For k = (2,3), $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and

$V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. A is con-s-k-$EP_3$

3 is con-s-k-EP_1 then

$AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not con-s-k-EP_1.

In this section, we explore the conditions for the product of con-s-k-EP_r matrices to be con-s-k-EP_r. Also we study the question of when BA is con-s-k-EP_r, for con-s-k-EP_r matrices A,B and AB.

Theorem 2.1

Let $A_1,A_2,A_3\ldots A_n(n\geq 1)$ be con-s-k-EP_r matrices and $A = A_1,A_2,A_3\ldots A_n$. Then the following statements are equivalent:

1. A is con-s-k-EP_r.
2. $R(A_I) = R(An)$ and $\rho(A) = r$
3. $R(A_i^T) = R(A_n^T)$ and $\rho(A) = r$.

Proof

Since A_1 and $A_n(n>1)$ are con-s-k-EP_r matrices, $R(A_1) = R(KVA_i^T)$ $R(A_n) = R(KVA_n^T)$ (by Theorem 2.). Since $A = A_1,A_2,A_3\ldots A_n$, $R(A) \subseteq R(A_1)$ and

$\rho(A) = \rho(A_1) \Rightarrow R(A) = R(A_1)$.

Also, $A^T = A_n^T A_{n-1}^T ....A_2^T A_1^T$, $R(A^T) \subseteq (RA_n^T)$ and $\rho(A^T) = \rho(A_n^T) = r \Rightarrow \rho(A^T) = \rho(A_n^T) = r$

Therefore,

$$R(A^T) = (RA_n^T) \Rightarrow R(KVA^T) = R(KVA_n^T).$$

Now, is con-s-k-EP$_r$ $\Leftrightarrow R(A) = R(KVA^T)$ and $\rho(A) = r$

$\Leftrightarrow R(A_1) = R(KVA_n^T)$ and $\rho(A) = r$

$\Leftrightarrow R(A_1) = R(A_n)$ and $\rho(A) = r$ (by Theorem 2.)

(ii) $\Leftrightarrow$ (iii)

$$R(A_1) = R(A_n) \Leftrightarrow R(KVA_i^T) = R(KVA_n^T)$$
$$\Leftrightarrow R(A_i^T) = R(A_n^T)$$

Hence the Theorem.

For the product of two con-s-k-EP$_r$ matrices A and B, Theorem (2.1) reduces to the following.

Corollary 2.2

Let A and B be con-s-k-EP$_r$ matrices, then AB is con-s-k-EP$_r$ $\Leftrightarrow \rho(AB) = r$ and $R(A) = R(B) \Leftrightarrow \rho(AB) = r$ and $R(A^T) = R(B^T)$.

Remark 2.3

In the above corollary (2.2) both the conditions that $\rho(AB) = r$ and R(A) = R(B) are essential for the product of two con-s-k-EP$_r$ matrices to be con-s-k-EP$_r$. This can be seen by the following example.

Example 2.4

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For k = (1) (2,3), the associated disjoint permutation matrix $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and

$$V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

A is con-s-k-EP_2 and B is con-s-k-EP_1

$$AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is not con-s-k-EP}_1$$

Here $\rho(AB) \neq 1$

Example 2.5

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For k = (1,2) (3) $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

$$V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

A is con-s-k-EP_3
B is con-s-k-EP_1 then

$$AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is not con-s-k-EP}_1.$$

Remark 2.6
If k(i) = i, for i=1,2,…..n, then corollary (2.2) reduces to the corollary for con-EP matrices.

Remark 2.7
Let $\rho(AB) = \rho(B) = r_1$, and $\rho(BA) = \rho(A) = r_2$, If AB, B are con-s-k-EP_r and A is con-s-k-EP_r then BA is con-s-k-EP_r.

Proof
Since, $\rho(BA) = \rho(A) = r_2$, to prove BA is con-s-k-EP_r, it is enough to show that $N(BA) = N[(BA)^T VK]$.

Now $N(A) \subseteq N(BA)$ and $\rho(BA) = \rho(A) \Rightarrow N(A) = N(BA)$

Also $N(B) \subseteq N(AB)$ and $\rho(AB) = \rho(B) \Rightarrow N(B) = N(AB)$

Now $N(BA) = N(A)$

$= N(A^T VK)$ (since A is con-s-k-EP)

$\subseteq N(B^T A^T VK)$

$= N((AB)^T VK)$

$= N(AB)$ (since A is con-s-k-EP)

$= N(B)$

$= N(B^T VK)$ (since A is con-s-k-EP)

$\subseteq N(A^T B^T VK)$

$= N((BA)^T VK)$

$N(BA) \subseteq N((BA)^T VK)$

Further, $\rho(BA) = \rho((BA^T)) = \rho((BA^T)VK)$

$\Rightarrow N(B(A)) = N(BA^T VK)$

Hence the Theorem

Lemma 2.8
If A,B are con-s-k-EP_r matrices and AB has rank r, then BA has rank r.

Proof
We know that [P.61][ ]

$\rho(AB) = \rho(A) - \dim(N(A) \cap N(B^T)^{\perp})$.

Since $\rho(AB) = \rho(A) = r, N(A) \cap N(B^T)^{\perp} = \{0\}$.

$N(A) \cap N(B^T)^{\perp} = \{0\} \Rightarrow N(A) \cap N(BVK)^{\perp}$

$= \{0\}$ (since B is con-s-k-EP_r)

$\Rightarrow N(A)^{\perp} \cap N(BVK) = \{0\}$.

$\Rightarrow N(A^T VK)^{\perp} \cap N(BVK) = \{0\}$ (since B is con-s-k-EP_r)

Now, $\rho(BA) = \rho((BVK)(KVA))$

$= \rho KVA - \dim(N(BVK) \cap N(A^T VK)^{\perp})$

$= \rho(KVA) - 0 = \rho(A) = r$

Hence the Lemma

Theorem 2.9
If A,B and AB are con-s-k-EP_r matrices then BA is con-s-k-EP_r.

Proof
Since A,B are con-s-k-EP_r matrices and $\rho(AB) = r$, by Lemma (2.8) $\rho(BA) = r$. Now the Theorem follows from Theorem (2.7) for $r_1 = r_2 = r$

Corollary 2.10
Let A,B be con-s-k-EP_r matrices. Then the following are equivalent.
I.AB is con-s-k-EP_r.
II.$(AB)^+$ is con-s-k-EP_r.
III.A^+B^+ is con-s-k-EP_r.
IV.B^+A^+ is con-s-k-EP_r.

Remark 2.11
In particular for k(i)=I, Theorem (2.9) reduces to the following.

Corollary
If A,B and AB are con-s-EP_r matrices, then BA is con-s-EP_r matrix.

References

[1] Anna Lee: Secondary symmetric, Secondary Skew symmetric, Secondary orthogonal matrices, period math, Hungary 7 (1976), 63-76.
[2] Ben Israel, A. and Greviue T.N.E: Generalized Inverses, Theory and Applications, Wiley, Newyour, (1974).
[3] Cantoni Anotnio and Bulter Paul., Properties of the Eigen vectors of persymmetric matrices with Applications to communication Theory, IEEE Trans, comm., 24 (1976), 804 – 809.
[4] Krishnamoorthy, S. Gunasekaran, K and Muthugobal,B.K.N., Con s-k-EP matrices, IJMSEA, Vol.5,No.1 (Jan 2011) to be published.
[5] Baksalary.J.K., Kala,R., and Klaczynski,K,1983, “The matrix inequalities M>B*MB”, Lin.Alg.Appl,54,77-86.