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Bounded Solutions for a System of Third-Order Nonlinear Neutral Delay Differential Equations

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Abstract

A system of third-order nonlinear neutral delay differential equations

$$\left[r_{12}(t)[r_{11}(t)(x_1(t)-P_1(t)x_1(t-\tau_1))']'\right]' = F_1(t, x_2(t-\sigma_1), \cdots, x_2(t-\sigma_m)),$$

$$\left[r_{22}(t)[r_{21}(t)(x_2(t)-P_2(t)x_2(t-\tau_2))']'\right]' = F_2(t, x_1(t-\sigma_1), \cdots, x_1(t-\sigma_m)),$$

where $\tau_i > 0, \sigma_1, \sigma_2, \cdots, \sigma_m \geq 0, r_{ij}(t) \in C([t_0, +\infty), \mathbb{R}^+), P_i(t) \in C([t_0, +\infty), \mathbb{R})$, $F_i \in C([t_0, +\infty) \times \mathbb{R}^m, \mathbb{R}), i, j \in \{1, 2\}$ is studied in this paper, and some sufficient conditions for existence of bounded solutions for this system are established by Krasnoselkii and Schauder fixed point theorems, and expressed through several theorems according to the range of the value of the functions $P_1(t), P_2(t)$ and their combination.

Keywords: Bounded solution, third-order neutral delay differential system, Krasnoselkii fixed point theorem, Schauder fixed point theorem.

MR(2000) Subject Classification: 34K15, 34C10.

Introduction and preliminaries

Recently, the interest in the study of differential equations and the system of differential equations has been increasing (see [1,3,6-14] and references cited therein).

We consider the following nonlinear differential system

$$\left[r_{12}(t)[r_{11}(t)(x_1(t)-P_1(t)x_1(t-\tau_1))']'\right]' = F_1(t, x_2(t-\sigma_1), \cdots, x_2(t-\sigma_m)),$$

$$\left[r_{22}(t)[r_{21}(t)(x_2(t)-P_2(t)x_2(t-\tau_2))']'\right]' = F_2(t, x_1(t-\sigma_1), \cdots, x_1(t-\sigma_m)),$$

which may be rewritten as

$$\begin{aligned}&\left[r_{i2}(t)[r_{i1}(t)(x_i(t)-P_i(t)x_i(t-\tau_i))']'\right]'\\&=F_i(t,x_{3-i}(t-\sigma_1),x_{3-i}(t-\sigma_2),\cdots,x_{3-i}(t-\sigma_m)),t\ge t_0,\end{aligned} \tag{1.1}$$

where $\tau_i>0,\sigma_1,\sigma_2,\cdots,\sigma_m\ge 0,r_{ij}(t)\in C([t_0,+\infty),\mathbb{R}^+),P_i(t)\in C([t_0,+\infty),\mathbb{R})$, $F_i\in C([t_0,+\infty)\times\mathbb{R}^m,\mathbb{R})$ and $i,j\in\{1,2\}$.

By applying Krasnoselkii and Schauder fixed point theorems, we obtained a few sufficient conditions for the existence of a bounded solution of the system (1.1).

Lemma 1.1(Krasnoselskii Fixed Point Theorem)[2] Let Ω be a bounded closed convex subset of a Banach space X and $Q,S:\Omega\to X$ satisfy $Qx+Sy\in\Omega$ for each $x,y\in\Omega$. If Q is a contraction mapping and S is a completely continuous mapping, then the equation $Qx+Sx=x$ has at least one solution in Ω.

Lemma 1.2(Schauder Fixed Point Theorem)[2] Let Ω be a closed, convex and nonempty subset of a Banach space X and $S:\Omega\to\Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is there exists an $x\in\Omega$ such that $Sx=x$.

Existence of bounded solutions

In this section, a few sufficient conditions of the existence of bounded solutions for system (1.1) will be given.

Theorem 2.1 Let functions $h_i,q_i,r_{ij}(t)\in C([t_0,+\infty),\mathbb{R}^+)$ and $P_i(t)\in C([t_0,+\infty),\mathbb{R})$ satisfy that

$$P_i(t)\equiv 1, \tag{2.1}$$

$$\left|F_i(t,u_1,u_2,\cdots,u_m)-F_i(t,v_1,v_2,\cdots,v_m)\right|\le h_i(t)\max\{|u_j-v_j|:1\le j\le m\}, \tag{2.2}$$

$$\left|F_i(t,u_1,u_2,\cdots,u_m)\right|\le q_i(t), \tag{2.3}$$

$$\int_{t_0}^{+\infty}s_1|R_{i1}{}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}{}'(s_2)|\int_{s_2}^{+\infty}\max\{q_i(s),h_i(s)\}ds\,ds_2\,ds_1<+\infty, \tag{2.4}$$

where $R_{ij}(t)=\int_{t_0}^{t}\frac{ds}{r_{ij}(s)}$ and $i,j\in\{1,2\}$. Then the system (1.1) has a bounded solution.

Proof According to a known result (Theorem 3.2.6 in [4]), (2.4) is equivalent to the condition

$$\sum_{k=0}^{\infty}\int_{t_0+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}\max\{q_i(s),h_i(s)\}dsds_2ds_1<+\infty,\quad i\in\{1,2\},\tag{2.5}$$

By (2.5), a sufficiently large $T>t_0$ can be chosen such that

$$\sum_{k=1}^{\infty}\int_{T+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}\max\{q_i(s),h_i(s)\}dsds_2ds_1<1,\quad i\in\{1,2\},\tag{2.6}$$

Let $C([t_0,+\infty),\mathbb{R}^2)$ be the set of all continuous vector functions $x(t)=(x_1(t),x_2(t))$ with the norm $\|x\|=\sup_{t\ge t_0}\{|x_1(t)|,|x_2(t)|\}<+\infty$. Obviously, $C([t_0,+\infty),\mathbb{R}^2)$ is a Banach space. Now, define a bounded, closed and convex subset Ω of $C([t_0,+\infty),\mathbb{R}^2)$ by:

$\Omega=\{x=(x_1,x_2)\in C([t_0,+\infty),\mathbb{R}^2):1\le x_i(t)\le 3, i\in\{1,2\}, t\ge t_0\}$.

Let mapping $S=(S_1,S_2):\Omega\to C([t_0,+\infty),\mathbb{R}^2)$ be defined by

$$(S_ix)(t)=\begin{cases}2+\sum_{k=1}^{\infty}\int_{t+k\tau_i}^{+\infty}R_{i1}'(s_1)\int_{s_1}^{+\infty}R_{i2}'(s_2)\int_{s_2}^{+\infty} & \\ \quad F_i(s,x_{3-i}(s-\sigma_1),\cdots,x_{3-i}(s-\sigma_m))dsds_2ds_1, & t\ge T\\ (S_ix)(T), & t_0\le t<T\end{cases}\tag{2.7}$$

for all $x\in\Omega$, where $i\in\{1,2\}$.

It is claimed that S is a self mapping on Ω. For all $x=(x_1,x_2)\in\Omega, i\in\{1,2\}$ and $t\ge T$, by (2.3) and (2.6), we have

$$(S_ix)(t)\ge 2-\sum_{k=1}^{\infty}\int_{T+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}q_i(s)dsds_2ds_1\ge 1,$$

$$(S_ix)(t)\le 2+\sum_{k=1}^{\infty}\int_{T+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}q_i(s)dsds_2ds_1\le 3.$$

Therefore, $S\Omega\subset\Omega$.

Now we show that S is continuous. Let $x_k=(x_{1k},x_{2k})\in\Omega$ and $x_{ik}(t)\to x_i(t)$ as $k\to+\infty$. Since Ω is closed, $x=(x_1,x_2)\in\Omega$. For $t\ge T$, (2.2) guarantees that

$$
\begin{aligned}
&|(S_i x_k)(t)-(S_i x)(t)| \\
&\le \sum_{k=1}^{\infty}\int_{t+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}|F_i(s,x_{3-i\,k}(s-\sigma_1),\cdots,x_{3-i\,k}(s-\sigma_m)) \\
&\quad -F_i(s,x_{3-i}(s-\sigma_1),\cdots,x_{3-i}(s-\sigma_m))|\,dsds_2ds_1 \\
&\le \sum_{k=1}^{\infty}\int_{t+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty} \\
&\quad h_i(\mathrm{s})\max\{|x_{3-i\,k}(s-\sigma_j)-x_{3-i}(s-\sigma_j):1\le j\le m|\,dsds_2ds_1 \\
&\le \|x_k-x\|\sum_{k=1}^{\infty}\int_{T+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}h_i(\mathrm{s})dsds_2ds_1
\end{aligned}
$$

This above inequality together with (2.6) implies that S is continuous.

Next, we prove $S\Omega$ is relatively compact. It is sufficient to show that the family of functions $\{Sx : x\in\Omega\}$ is uniformly bounded and equicontinuous on $[t_0,+\infty)$. $S\Omega\subset\Omega$ ensures the uniform boundedness. For the equicontinuity, it is only need to prove that, for any given $\varepsilon>0$, $[t_0,+\infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ε. By (2.6), for any $\varepsilon>0$, take $T'\ge T$ large enough so that

$$
\sum_{k=1}^{\infty}\int_{T'+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}q_i(s)dsds_2ds_1<\frac{\varepsilon}{2}. \tag{2.8}
$$

Then, for any $x\in\Omega$ and $t_2>t_1\ge T'$, (2.8) ensures that

$$
\begin{aligned}
&|(S_i x)(t_2)-(S_i x)(t_1)| \\
&\le \sum_{k=1}^{\infty}\int_{t_2+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}|F_i(s,x_{3-i}(s-\sigma_1),\cdots,x_{3-i}(s-\sigma_m))|dsds_2ds_1 \\
&\quad +\sum_{k=1}^{\infty}\int_{t_1+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}|F_i(s,x_{3-i}(s-\sigma_1),\cdots,x_{3-i}(s-\sigma_m))|dsds_2ds_1 \\
&\le \sum_{k=1}^{\infty}\int_{T'+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}q_i(\mathrm{s})dsds_2ds_1 \\
&\quad +\sum_{k=1}^{\infty}\int_{T'+k\tau_i}^{+\infty}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}q_i(\mathrm{s})dsds_2ds_1 \\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

For any $x\in\Omega$ and $T\le t_1<t_2\le T'$, there exists $\delta>0$ such that if $0<t_2-t_1<\delta$, then

$$
\begin{aligned}
&|(S_i x)(t_2)-(S_i x)(t_1)| \\
&\le \sum_{k=1}^{\infty}\int_{t_1+k\tau_i}^{t_2+k\tau_i}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}|F_i(s,x_{3-i}(s-\sigma_1),\cdots,x_{3-i}(s-\sigma_m))|dsds_2ds_1 \\
&\le \sum_{k=1}^{\infty}\int_{t_1+k\tau_i}^{t_2+k\tau_i}|R_{i1}'(s_1)|\int_{s_1}^{+\infty}|R_{i2}'(s_2)|\int_{s_2}^{+\infty}q_i(s)dsds_2ds_1<\varepsilon.
\end{aligned}
$$

For any $x\in\Omega$ and $t_0\le t_1<t_2\le T$, it is easy to get that
$|(S_i x)(t_2)-(S_i x)(t_1)|=0<\varepsilon.$

Consequently, $\{S_i x: x\in\Omega\}$ is uniformly bounded and equicontinuous on $[t_0,+\infty)$. Therefore $S\Omega$ is relatively compact. Applying Lemma 1.2, we could find a $x_0=(x_{01},x_{02})\in\Omega$ such that $Sx_0=x_0$. That is

$$
x_{0i}(t)=\begin{cases} 2+\sum_{k=1}^{\infty}\int_{t+k\tau_i}^{+\infty}R_{i1}'(s_1)\int_{s_1}^{+\infty}R_{i2}'(s_2)\int_{s_2}^{+\infty} & \\ \quad F_i(s,x_{0\ 3-i}(s-\sigma_1),\cdots,x_{0\ 3-i}(s-\sigma_m))dsds_2ds_1, & t\ge T \\ x_{0i}(T), & t_0\le t<T \end{cases} \tag{2.9}
$$

where $i\in\{1,2\}$. For $t\ge T$,

$$
\begin{aligned}
x_{0i}(t)-x_{0i}(t-\tau_i)=-\int_t^{+\infty}R_{i1}'(s_1)\int_{s_1}^{+\infty}R_{i2}'(s_2)\int_{s_2}^{+\infty} \\
F_i(s,x_{0\ 3-i}(s-\sigma_1),\cdots,x_{0\ 3-i}(s-\sigma_m))dsds_2ds_1.
\end{aligned}
$$

Then,

$$
(x_{0i}(t)-x_{0i}(t-\tau_i))'=R_{i1}'(t)\int_t^{+\infty}R_{i2}'(s_2)\int_{s_2}^{+\infty}F_i(s,x_{0\ 3-i}(s-\sigma_1),\cdots,x_{0\ 3-i}(s-\sigma_m))dsds_2,
$$

which we can rewrite it as

$$
r_{i1}(t)(x_{0i}(t)-x_{0i}(t-\tau_i))'=\int_t^{+\infty}R_{i2}'(s_2)\int_{s_2}^{+\infty}F_i(s,x_{0\ 3-i}(s-\sigma_1),\cdots,x_{0\ 3-i}(s-\sigma_m))dsds_2.
$$

Finding the derivative,

$$
[r_{i1}(t)(x_{0i}(t)-x_{0i}(t-\tau_i))']'=-R_{i2}'(t)\int_t^{+\infty}F_i(s,x_{0\ 3-i}(s-\sigma_1),\cdots,x_{0\ 3-i}(s-\sigma_m))ds.
$$

Proceeding as before, we get

$$
\left[r_{i2}(t)[r_{i1}(t)(x_{0i}(t)-x_{0i}(t-\tau_i))']'\right]'=F_i(t,x_{0\ 3-i}(t-\sigma_1),\cdots,x_{0\ 3-i}(t-\sigma_m)).
$$

Therefore, $x_0(t)$ is a bounded solution of the system (1.1). This completes the proof.

Theorem 2.2 Let functions $h_i, q_i, r_{ij}(t) \in C([t_0,+\infty),\mathbb{R}^+)$ and $P_i(t) \in C([t_0,+\infty),\mathbb{R})$ satisfy that (2.2), (2.3) and

$$|P_i(t)| \le \overline{P}_i < \frac{1}{2}, \tag{2.10}$$

$$\int_{t_0}^{+\infty} \max\{\frac{1}{|r_{ij}(t)|}, q_i(t), h_i(t)\} dt < +\infty, \tag{2.11}$$

where $i, j \in \{1,2\}$. Then the system (1.1) has a bounded solution.

Proof: In virtue of (2.11), a sufficiently large $T > t_0$ can be chosen such that

$$\int_T^{+\infty}\int_{s_1}^{+\infty}\int_{s_2}^{+\infty} \frac{q_i(s)}{|r_{i1}(s_1)r_{i2}(s_2)|} ds ds_2 ds_1 \le \frac{1}{2} - \overline{P}_i, \tag{2.12}$$

where $i \in \{1,2\}$. Let $C([t_0,+\infty),\mathbb{R}^2)$ be the set like that in the proof of Theorem 2.1 and define a bounded, closed and convex subset Ω of $C([t_0,+\infty),\mathbb{R}^2)$ as following:

$$\Omega = \{x = (x_1, x_2) \in C([t_0,+\infty),\mathbb{R}^2) : 0 \le x_i(t) \le 1, i \in \{1,2\}, t \ge t_0\}.$$

Let mappings $Q = (Q_1, Q_2)$ and $S = (S_1, S_2) : \Omega \to C([t_0,+\infty),\mathbb{R}^2)$ be defined by

$$(Q_i x)(t) = \begin{cases} \frac{1}{2} + P_i(t)x(t-\tau_i), & t \ge T \\ (Q_i x)(T), & t_0 \le t < T \end{cases} \tag{2.13}$$

$$(S_i x)(t) = \begin{cases} -\int_t^{+\infty}\int_{s_1}^{+\infty}\int_{s_2}^{+\infty} \frac{F_i(s, x_{3-i}(s-\sigma_1),\cdots,x_{3-i}(s-\sigma_m))}{r_{i1}(s_1)r_{i2}(s_2)} ds ds_2 ds_1, & t \ge T \\ (S_i x)(T), & t_0 \le t < T \end{cases} \tag{2.14}$$

for all $x \in \Omega$, where $i \in \{1,2\}$.

(i) It is claimed that $Qx + Sy \in \Omega$ for all $x, y \in \Omega$, i.e. $Q\Omega \cup A\Omega \subset \Omega$.

In fact, for each $x, y \in \Omega$ and $t \ge T$, it follows from (2.3), (2.10) and (2.12) that

$$\begin{aligned} &(Q_i x)(t) + (S_i y)(t) \\ &\ge \frac{1}{2} - \overline{P}_i x(t-\tau_i) - \int_t^{+\infty}\int_{s_1}^{+\infty}\int_{s_2}^{+\infty} \left| \frac{F_i(s, y_{3-i}(s-\sigma_1),\cdots,y_{3-i}(s-\sigma_m))}{r_{i1}(s_1)r_{i2}(s_2)} \right| ds ds_2 ds_1 \\ &\ge \frac{1}{2} - \overline{P}_i - \int_T^{+\infty}\int_{s_1}^{+\infty}\int_{s_2}^{+\infty} \frac{q_i(s)}{|r_{i1}(s_1)r_{i2}(s_2)|} ds ds_2 ds_1 \ge 0 \end{aligned}$$

and

$$(Q_i x)(t)+(S_i y)(t)\le \frac{1}{2}+\overline{P_i}+\int_T^{+\infty}\int_{s_1}^{+\infty}\int_{s_2}^{+\infty}\frac{q_i(s)}{|r_{i1}(s_1)r_{i2}(s_2)|}dsds_2ds_1\le 1.$$

Thus, $0\le (Q_i x)(t)+(S_i y)(t)\le 1, i\in\{1,2\}$ for $t>t_0$.

(ii) It is declared that Q is a contraction mapping on Ω.

In reality, for any $x,y\in\Omega$ and $t\ge T$, it is easy to derive that

$$|(Q_i x)(t)+(Q_i y)(t)|\le |P_i(t)|\,|x(t-\tau_i)-y(t-\tau_i)|\le \overline{P_i}\,\|x-y\|\le \frac{1}{2}\|x-y\|,$$

which implies that

$$\|Q_i x-Q_i y\|\le \frac{1}{2}\|x-y\|.$$

That is, Q is a contraction mapping on Ω.

(iii) It can be asserted that S is completely continuous, just like what we did in Theorem 2.1. Hence, we omit it.

It follows from Lemma 1.1 that there is $x_0\in\Omega$ such that $Qx_0+Sx_0=x_0$. Obviously, $x_0(t)$ is a bounded solution of the system (1.1). This completes the proof.

Theorem 2.3 Let functions $h_i,q_i,r_{ij}(t)\in C([t_0,+\infty),\mathbb{R}^+)$ and $P_i(t)\in C([t_0,+\infty),\mathbb{R})$ satisfy that (2.2), (2.3) and

$$P_1(t)\equiv 1, \tag{2.15}$$

$$|P_2(t)|\le \overline{P_2}<\frac{1}{2}, \tag{2.16}$$

$$\int_{t_0}^{+\infty}s_1|R_{11}'(s_1)|\int_{s_1}^{+\infty}|R_{12}'(s_2)|\int_{s_2}^{+\infty}\max\{q_1(s),h_1(s)\}dsds_2ds_1<+\infty, \tag{2.17}$$

$$\int_{t_0}^{+\infty}\max\{\frac{1}{|r_{2j}(t)|},q_2(t),h_2(t)\}dt<+\infty, \tag{2.18}$$

where $j\in\{1,2\}$. Then the system (1.1) has a bounded solution.

Proof By (2.17) and (2.18), a sufficiently large $T>t_0$ can be chosen such that

$$\sum_{k=1}^{\infty}\int_{T+k\tau_i}^{+\infty}|R_{11}'(s_1)|\int_{s_1}^{+\infty}|R_{12}'(s_2)|\int_{s_2}^{+\infty}\max\{q_1(s),h_1(s)\}dsds_2ds_1<1, \tag{2.19}$$

$$\int_T^{+\infty}\int_{s_1}^{+\infty}\int_{s_2}^{+\infty}\frac{\max\{q_2(s),h_2(s)\}}{|r_{21}(s_1)r_{22}(s_2)|}dsds_2ds_1\le\frac{1}{2}-\overline{P_i},\tag{2.20}$$

where $i\in\{1,2\}$. Let $C([t_0,+\infty),\mathbb{R}^2)$ be the set like that in the proof of Theorem 2.1 and define a bounded, closed and convex subset Ω of $C([t_0,+\infty),\mathbb{R}^2)$ as following:

$$\Omega=\{x=(x_1,x_2)\in C([t_0,+\infty),\mathbb{R}^2):1\le x_1(t)\le 3,0\le x_2(t)\le 1,t\ge t_0\}.$$

Let mappings S_1,Q_2 and $S_2:\Omega\to C([t_0,+\infty),\mathbb{R}^2)$ be defined as

$$(S_1x)(t)=\begin{cases}2+\sum_{k=1}^{\infty}\int_{t+k\tau_1}^{+\infty}R'_{11}(s_1)\int_{s_1}^{+\infty}R'_{12}(s_2)\int_{s_2}^{+\infty}\\ \quad F_1(s,x_2(s-\sigma_1),...,x_2(s-\sigma_m))dsds_2ds_1, & t\ge T\\ (S_1x)(T), & t_0\le t<T\end{cases}\tag{2.21}$$

$$(Q_2x)(t)=\begin{cases}\frac{1}{2}+P_2(t)x(t-\tau_2), & t\ge T\\ (Q_2x)(T), & t_0\le t<T\end{cases}\tag{2.22}$$

$$(S_2x)(t)=\begin{cases}-\int_t^{+\infty}\int_{s_1}^{+\infty}\int_{s_2}^{+\infty}\frac{F_2(s,x_1(s-\sigma_1),...,x_1(s-\sigma_m))}{r_{21}(s_1)r_{22}(s_2)}dsds_2ds_1, & t\ge T\\ (S_1x)(T), & t_0\le t<T\end{cases}\tag{2.23}$$

for all $x\in\Omega$.

Proceeding similarly as in the proof of Theorem 2.1 and 2.2, we get that there are $x_{01},x_{02}\in\Omega$ such that $S_1x_{01}=x_{01}$ and $Q_2x_{02}+S_2x_{02}=x_{02}$. Then $x_0(t)=(x_{01}(t),x_{02}(t))$ is a bounded solution of the system (1.1). This finishes the proof.

Remark 2.4 Proceeding as before, we can prove that no matter $P_i(t)$ belongs to which cases:

(1) $P_i(t) \equiv 1,$

(2) $|P_i(t)| \le \overline{P_i} < \frac{1}{2},$

(3) $0 < P_i(t) \le \overline{P_i} < 1,$

(4) $1 < \underline{P_i} \le P_i(t) \le \overline{P_i} < +\infty,$

(5) $-1 < \underline{P_i} \le P_i(t) < 0,$

(6) $-\infty < \underline{P_i} \le P_i(t) \le \overline{P_i} < -1,$

(7) any combination of the above.

The system (1.1) has a bounded solution.

References

[1] J.F. Cheng, Z. Annie, Existence of nonoscillatory solution to second order linear neutral delay equation, J. Sys. Sci. Math. Scis. 24(3) (2004), 389–397

[2] J. Dugundji and A. Granas, Fixed Point Theory, Springer-Verlag, New York (2003)

[3] H. El-Metwally, M.R.S. Kulenovic and S. Hadziomerspahic, Nonoscillatory solutions for system of neutral delay equation, Nonlinear Anal. 54 (2003), 63–81

[4] L.H. Erbe, Q.K. Kong and B.G. Zhang, Oscillation theory for functional differential equations, Marcel Dekker, New York (1995)

[5] J.K. Hale, Theory of functional differential equatin, Springer-Verlag, New York (1977)

[6] J. Shen and X. Tang, New nonoscillation criteria for delay differential equations, J. Math. Anal. Appl. 290 (2004), 1–9

[7] Y. Yu and H. Wang, Nonoscillatory solutions of second-order nonlinear neutral delay equations, J. Math. Anal. Appl. 311 (2005), 445–456

[8] Zhenyu Guo, The solvability of a new higher-order nonlinear neutral delay difference equation, Internat. J. Pure Appl. Math. 57(1) (2009) 69-76

[9] Zhenyu Guo, The solvability of a higher-order nonlinear neutral delay difference equation, Vietnam J. Math. 38(1) (2010) 79-87

[10] Min Liu and Zhenyu Guo, Solvability of a Higher-Order Nonlinear Neutral Delay Difference Equation, Advances in Difference Equations doi:10.1155/2010/767620 14pages [SCIE]

[11] Zhenyu Guo and Min Liu, Existence of non-oscillatory solutions for a higher-order nonlinear neutral difference equation, Electronic Journal of Differential Equations Vol. 2010, No. 146, pp. 1-7

[12] Zhenyu Guo and Min Liu, Unique solutions for systems of fractional order differential equations with infinite delay, Bulletin of Mathematical Analysis and Applications 3(1) (2011) 142-147

[13] Zhenyu Guo, Xiaoying Zhao and Min Liu, Nonosillatory solutions of a higher-order nonlinear neutral delay differential equation, Internat. J. Pure Appl. Math. 70(1) (2011) 1-10

[14] Zhenyu Guo, Existence of Nonoscillatory Solutions for a Second-Order Nonlinear Neutral Delay Differential Equation, Journal of Mathematical Research & Exposition 31(3) (2011) 503–508