Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 7, Number 3 (2011), pp. 237-243

http://www.ripublication.com/gjpam.htm

Set-valuations of a Digraph

B.D. Acharya[1], K.A. Germina[2] and Jisha Elizabeth Joy[3]

[1]Srinivasa Ramanujan Center for Intensification of Interaction in Interdisciplinary Discrete Mathematical Sciences (SRC-IIIDMS), University of Mysore, Mysore - 570005, India
E-mail: devadas.acharya@gmail.com
[2,3]Mathematics Research Center, Mary Matha Arts & Science College, Vemom P.O., Mananthavady-670645, Kerala, India
E-mail: srgerminaka@gmail.com, jishaelizabethjoy@gmail.com

Abstract

Given a simple digraph D = (V, *A*) and a set-valuation f: V $\to 2^X$ to each arc (u,v) of D we assign the set f(u) - f(v). A set-valuation f of a given digraph D = (V, *A*) is a set-indexer of D if both f and its `arc-induced function' g_f, defined by letting g_f(u,v) = f(u) - f(v) for each arc (u,v) of D, are injective. Further, f is arc-bounded if $|g_f(u,v)| < |f(v)|$, for each (u,v) in *A*. The aim of this paper is to report results of our preliminary analysis of these notions and generate new ideas for advanced studies in specific directions, indicated through open problems and conjectures.

Keywords: digraph, set-valuation, set-indexer, set-graceful, set-sequential, arc-binding

Mathematics Subject Classification: 05C 10, 78.

Introduction

Unless mentioned or defined otherwise, for all the terminology and notation in graph theory the reader is referred to [2, 4]. Also, unless stated otherwise, all graphs and digraphs considered in this paper are finite, simple and without self-loops.

By defining an assignment of subsets of a `ground set' to the vertices of a graph G and the symmetric difference of subsets of X associated with end vertices of an edge to the corresponding edge in G, calling it a *set-valuation* of G in general, Acharya [1] gave a new direction to the very topic of graph labeling. For a (p,q)-graph G = (V, E) and a nonempty set X of cardinality n, he defined a *set-indexer* of G as an injective `set-valued' function (or, equivalently, `set-labeling') *f*: V $\rightarrow 2^X$ such that the induced `edge-function' $f^{\oplus}: E \rightarrow 2^X - \varphi$ defined by $f^{\oplus}(u,v) = f(u) \oplus f(v)$ for all $(u,v) \in E(G)$ is also injective, where 2^X is the set of all subsets of X and $\oplus$ denotes the binary operation of taking the symmetric difference of pairs of subsets of X. He called a graph G, not necessarily finite, *set-graceful* if it admits a *set-graceful labeling* (or, *graceful set-indexer*) *f* which is defined as a set-indexer such that $f^{\oplus}(E(G)) = \{f^{\oplus}(u,v): (u,v) \in E(G)\} = 2^X - \varphi$. Also, he defined G to be set-sequential if G admits a *set-sequential labeling*, which is a bijection $f: V \cup E \rightarrow 2^X - \varphi$ such that $f^{\oplus}(u,v) = f(u) \oplus f(v)$ for all $(u,v) \in E(G)$.

Further, Acharya [1] showed that every graph possesses a set-indexer $f: V \rightarrow 2^X$ such that $f(V) = \{f(v): v \in V(G)\}$ is a topology on the ground set X; he called such a set-indexer *a topological set-indexer* (or, a *T-set-indexer*) of G. In particular, a set-graceful graph G is *topologically set-graceful* (or, `*T-set-graceful'*) if G admits a graceful set-indexer *f* such that *f(V)* is a topology on X; such set-indexers are called *graceful topological set-indexers* of G. Hence, he defined the *topological number* (or, simply, the *T-number*) t(G) of G as the smallest cardinality of a ground set X with respect to which G has a T-set-indexer.

Notice that given any set-valuation *f* of a graph G, one can construct a set-valued symmetric digraph $(\vec{G}, f)$ by splitting every edge uv of G into two arcs (u,v) and (v,u) assigned with the set-labels $f(u) - f(v)$ and $f(v) - f(u)$ respectively. We get back the original set-labeled graph (*G*,*f*) from $(\vec{G}, f)$ by merging every symmetric arc (u,v), (v,u) into the undirected edge uv together with $(f(u) - f(v)) \cup (f(v) - f(u)) = f(u) \oplus f(v)$ as its set-label. This motivates the new idea to generalize the very notion of set-valuations to arbitrary simple digraphs:

Definition 1.1: Given a simple digraph D = (V, *A*) and a set-valuation $f: V \rightarrow 2^X$, to each arc (u,v) in D assign $f(u) - f(v)$. A set-valuation *f* of a given digraph D = (V, *A*) is a *set-indexer* of D if both *f* and its `edge-induced function' g_f defined by letting $g_f(u,v) = f(u) - f(v)$ for each arc (u,v) of D, are injective. Further, let us call *f arc-binding* if $|g_f| < |f(v)|$ for each (u,v) in *A*.

For example, the single-arc digraph u $\rightarrow$ v has an arc-binding set- indexer *f* defined by $f(u) = \varphi$ and $f(v) = \{1\}$.

As a second example, take the 3-dicycle Z_3 = (u,v,w,u). Then, it has an arc-binding set-indexer *f* defined by letting $f(u) = \{1,3\}$, $f(v) = \{1,2\}$, $f(w) = \{1,2,3\}$. Corresponding non-examples are obtained by reversing the labels of the vertices in the first example and, for Z_3 in the second example, label u,v,w respectively by the sets φ, {3} and {1,2}. That is, in each case, the set-labeling is not arc-binding.

The aim of this article is to report results of our preliminary analysis of the above notions and generate new ideas for advanced studies in specific directions, indicated through open problems and conjectures.

Set-indexed digraphs

Observation 2.1: Let f: V $\rightarrow 2^X$,be any set-valuation of a digraph D = (V, *A*). If (u,v) is an arc of D such that f(u) $\subseteq$ f(v) then g_f (u,v) = φ.

Observation 2.2: If D = (V, *A*) is a set-indexed digraph, then any vertex *v* can have at most one successor *u* such that f(u) $\cap$ f(v) = φ.

Proof: Suppose under the hypothesis that there are two successors u, w of v such that f(u) $\cap$ f(v) = φ and f(w) $\cap$ f(v) = φ. Then, g_f (u, v) = f(u) - f(v) = f(v) = f(v) – f(w) = g_f (v, w) which is a contradiction to the injectivity of g_f.

Lemma 2.3: The out-degree *od(v)* of any vertex u in a set-indexed digraph (*D, f*) is at most $2^{|f(v)|}$

Proof: Let D = (V, *A*) be a set-indexed digraph with a set-indexer f: V $\rightarrow 2^X$. Let f(v) = $X_1 \in 2^X$. Since f is a set-indexer of D, for an arc (v, v_j) in D, g_f (v, v_j) = f(v) - f(v_j) should necessarily be in one of the sets S_0, S_1 , …, $S_{|X1|-1}$ where S_i are sets of subsets of X_1 with cardinality *i*. Then, $\sum_{i=0}^{|X_1|-1} S_i = 2^{|X_1|} - 1$ as there cannot be a successor (or, equivalently, an `out-neighbor') of v labeled X_1 due to injectivity of f. Moreover, there could be at most one successor of v whose label is a super set of X_1 without violating the injectivity of the induced arc function g_f. Thus, od(v) $\leqslant 2^{X1} = 2^{|f(v)|}$

Theorem 2.4: Every digraph admits a set-indexer.

Proof: Let D = (V, *A*) be a digraph where V = {v_1, v_2, …, v_n} and A = {$e_{ij} = v_iv_j$: v_i, v_j $\in$ V}. Choose X = V U A as the ground set. Define a function f: V $\rightarrow 2^X$ by f(v_i) = {v_i} U {$e_{ij} \in$ A| e_{ij} is incident with v_i} and the induced arc function is g_f (u, v) = f(u) - f(v). Injectivity of f follows from the definition of f. Now, consider any two arcs $e_{ij} = v_iv_j$ and $e_{lm} = v_lv_m$. If $v_i \neq v_l$, then g_f (e_{ij}) $\neq$ g_f (e_{lm}) as $v_i \in g_f$ (e_{ij}) and $v_i \notin g_f$ (e_{lm}). If $v_i = v_l$ then, $e_{ij} = e_{lm} = e_{im} = v_iv_m$ which in turn implies $e_{ij} \in g_f$ (e_{im}) ; but $e_{ij} \notin g_f$ (e_{ij}). That is, g_f (e_{im}) $\neq$ g_f (e_{ij}) and hence g_f is injective.

The following problem is open.

Problem 1: What is the least cardinality of a ground set X with respect to which a given digraph D admits a set-indexer?

If ω′ (D)denotes the least cardinality of a ground set X with respect to which the given digraph D admits a set-indexer, then from the proof of Theorem 2. 4 one can infer that

$$\omega'(D) \leq |V(D)| \text{ U } |A(D)| \qquad (2)$$

It would be interesting to determine digraphs D for which equality is attained in (2). The question whether the bound in (2) could be improved in general is open.

Arc-binding set-indexers of a digraph

Note that any set-indexer f of a digraph D = (V, A) has the property that $|g_f(u,v)| = |f(u) - f(v)| \leq |f(u)|$ for any arc (u,v) but $|g_f(u,v)|$ could be equal to, or larger than, $|f(v)|$. This motivates the following definition.

Definition 3.1: A set-indexer f of a given digraph D = (V, A) is *arc-binding* if $|g_f(u,v)| \leq |f(v)|$ for each (u,v) in A.

Theorem 3.2: Every digraph admits an arc-binding set-indexer.

Proof: Let D = (V, A) be a digraph where V = $\{v_1, v_2, \ldots, v_n\}$.

When $n = 1$, let $f(v_1) = \{1\}$. Clearly, f is an arc-binding set-indexer.

When $n = 2$, let X = $\{0, v_1, v_2\}$. Let $f: V \rightarrow 2^X$ be defined by $f(v_i) = \{0, v_i\}$, $i = 1, 2$. Then, the induced arc function g_f yields $g_f(v_i, v_j) = f(v_i) - f(v_j) = \{0, v_i\} - \{0, v_j\} = \{v_i\}$. Now, $|g_f(v_i, v_j)| = |f(v_i) - f(v_j)| = 1 < |f(v_j)|$. Therefore, f is an arc-binding set-indexer.

Next, let $n \geq 3$. Let $X_1 = V$ and $X_2 = \{1, 2, \ldots, n\}$. Choose $X = X_1 \cup X_2$ as the ground set. Let $f: V \rightarrow 2^X$ be defined by $f(v_i) = \{v_i\} \cup (X_2 - \{i\})\}$ where $1 \leq i \leq n$. Then, the induced arc function g_f yields $g_f(v_i, v_j) = f(v_i) - f(v_j) = \{v_i\} \cup (X_2 - \{i\})\} - \{v_j\} \cup (X_2 - \{j\})\} = \{v_i, j\}$. Injectivity of f follows from the definition of f. Now, consider any two arcs $e_{ij} = v_iv_j$ and $e_{lm} = v_lv_m$. If $v_i \neq v_l$, then $g_f(e_{ij}) \neq g_f(e_{lm})$ as $v_i \in g_f(e_{ij})$ and $v_i \notin g_f(e_{ij})$. If $v_i = v_l$ then, $e_{ij} = e_{lm} = e_{im} = v_iv_m$ which in turn implies $e_{ij} \in g_f(e_{im})$ but $e_{ij} \notin g_f(e_{ij})$. That is, $g_f(e_{im}) \neq g_f(e_{ij})$ and hence g_f is injective. Further, $|f(v_j)| = |\{v_j\} \cup (X_2 - \{j\})\}| = 1 + n - 1 = n$ and $|f(v_i) - f(v_j)| = |\{v_i, j\}| = 2$. This implies $|g_f(v_i, v_j)| = |f(v_i) - f(v_j)| = 2 < |f(v_j)| = n$ for every i, j. Therefore, f is an arc-binding set-indexer of D.

Remark 3.3: Note that the proof of Theorem 3.2 indicates that it is valid even if D is an infinite digraph.

Thus, for any given digraph D = (V, A) and for any arc-binding set-indexer f we have

$$|g_f(u, v)| \leq \min \{|f(u)|, |f(v)|-1\}, \text{ for every } (u, v) \in A \quad (3)$$

The following problem arises.

Problem 2: Characterize digraphs D = (V, A) that admit arc-binding set-indexers f such that $|g_f(u, v)| = \min \{|f(u)|, |f(v)|-1\}$, for every $(u, v) \in A$ (4)

The following problem is open.

Problem 3: What is the least cardinality of a ground set X with respect to which a given digraph D admits an arc-binding set-indexer?

If ω^{ab} (D)denotes the least cardinality of a ground set X with respect to which the given digraph D admits an arc-binding set-indexer then from the proof of Theorem 3.2 one can infer that

$$\omega^{ab}(D) \leq 2n \tag{5}$$

where n = |V(D)|. It would be interesting to determine digraphs D for which equality is attained in (5). Again, the question whether the bound in (5) could be improved in general is open. In any case, we have

$$\omega'(D) \leq \omega^{ab}(D) \tag{6}$$

for any digraph D.

Problem 4: Characterize digraphs D for which equality holds in (6).

For digraphs D of order n, the inequalities (5) and (6) can be put together as

$$\omega'(D) \leq \omega^{ab}(D) \leq 2n \tag{7}$$

Another problem that naturally arises is the following.

Problem 5: Find a 'good' lower bound for ω' (D).

For attempting to answer some of the above questions, one needs to look at fundamental properties of (arc-binding) set-indexers of a digraph, which we shall hence pursue.

Proposition 3.4: Let *f:* V $\rightarrow 2^X$ be any arc-binding set-indexer of a digraph D = (V, A) and u be a vertex in D such that *f*(u) = φ. Then, u is a source with od(u) = 1.

Proof: Let *f* be an arc-binding set-indexer of D. If possible, let *u* be not a source of D. Then there must exist $a \in$ V(D) such that (a, u) $\in$ A and $|g_f(a, u)| = |f(a) - f(u)| = |f(a)|$, because f(u) = φ by hypothesis. But then, invoking injectivity of *f*, we get $|g_f(a, u)| = |f(a)| > 0 = |f(u)|$, a contradiction to the hypothesis that *f* is an arc-binding set-indexer of D. Therefore, *u* is a source in D. Now, invoking Observation 2.2 we conclude that *od(u)* cannot be greater than one and hence *od(u) = 1*. Hence, *u* is a source with *od(u) = 1*.

Lemma 3.5.: Let *f:* V→ 2^X be any arc-binding set-indexer of a digraph D = (V, *A*). Then, for any arc (u, v) in D with |*f*(u)| = |*f*(v)|, one has *f*(u) ∩ *f*(u) = φ.

Proof: Let *f:* V→ 2^X be any arc-binding set-indexer of a digraph D = (V, A) with an arc (u, v) such that |*f*(u)| = |*f*(v)|. If possible, suppose *f*(u) ∩ *f*(u) $\neq \phi$. Then, $|g_f(u, v)| = |f(u) - f(v)| = |f(u)| = |f(v)|$, a contradiction to the hypothesis that *f* is arc-binding.

Lemma 3.6: If (u, v) is a symmetric arc in a digraph D = (V, A) together with an arc-binding set-indexer f then,(i) $f(u) \cap f(u) \neq \varphi$, and (ii) $|f(u)| > 1$ and $|f(v)| > 1$.

Proof: (i) If possible, under the hypothesis, suppose $f(u) \cap f(u) = \varphi$. Then, we have either $|f(u)| = |f(v)|$ or one of these numbers is less than the other. If $|f(v)| = |f(u)|$ then by Lemma 3.5, $f(u) \cap f(v) \neq \phi$. So without loss of generality let $|f(u)| < |f(v)|$. Then , $|g_f(u, v)| = |f(u) - f(v)| = |f(u)| < |f(v)|$. But $|g_f(v, u)| = |f(v) - f(u)| = |f(v)| > |f(u)|$ a contradiction to our assumption. Hence, $f(u) \cap f(v) \neq \varphi$.

(ii) Suppose $|f(v)| \leq 1$. That is, either $|f(v)|=0$ or $|f(v)|=1$. If $|f(v)|=0$ then v is a source, a contradiction to the hypothesis that (u, v) is a symmetric arc. Now, $|f(v)|= 1$ is not possible by (i). Hence, $|f(v)|>1$. Similarly, we can prove, $|f(u)| > 1$.

In cases of specific digraphs D, one can find arc-binding set-indexers with ground sets X having much lower cardinalities than the one found in the general case as in the proof of Theorem 3.2, thereby improving the bound given by (5). In what follows, we give some examples of this kind.

Theorem 3.7: For the directed path $\overrightarrow{P_n}$, $\omega^{ab}(\overrightarrow{P_n}) \leq n$.

Proof: Let $\overrightarrow{P_n}$ be the directed path of order n with $V(\overrightarrow{P_n}) = \{v_1, v_2, \ldots, v_n\}$. Choose X = {1, 2, …, n} and define $f: V(\overrightarrow{P_n}) \to 2^X$ by letting $f(v_i) = \{1, i + 1\}$, $1 \leq i \leq n$. Clearly f is injective. Further, $g_f(v_i, v_{i+1}) = f(v_i) - f(v_{i+1}) = \{1, i+1\} - \{1, i+2\} = \{i+1\}$, $1 \leq i \leq n$. Clearly g_f is also injective. Moreover, $|g_f(v_i, v_{i+1})| = |\{i+1\}| = 1 < |f(v_i)|$. Therefore, f is an arc-binding set-indexer of $\overrightarrow{P_n}$ and the result follows.

Theorem 3.8: For the directed cycle $\overrightarrow{Z_n}$, $n \geq 3$, $\omega^{ab}(\overrightarrow{Z_n}) \leq n + 1$.

Proof: Let $V(\overrightarrow{Z_n}) = \{v_1, v_2, \ldots, v_n\}$. Choose X = {1, 2,…, n +1} as the ground set and define $f: V(\overrightarrow{Z_n}) \to 2^X$ such that $f(v_i) = \{1, i +1\}$, $1 \leq i \leq n-1$ and $f(v_n) = \{1, 2, n+1\}$. Clearly, f is injective. Further, $g_f(v_i, v_{i+1}) = \{i +1\}$, $1 \leq i \leq n-1$ and $g_f(v_n, v_1) = \{n+1\}$. Therefore, g_f is also injective. Moreover, $|g_f(v_i, v_{i+1})| = |\{i +1\}| = 1 < |f(v_i)|$ for all i. Thus, f is an arc-binding set-indexer of $\overrightarrow{Z_n}$.

Theorem 3.9: For the directed star $\overrightarrow{K_{1,n}}$, $\omega^{ab}(\overrightarrow{K_{1,n}}) \leq n + 2$.

Proof: The proofs for n = 1 and n = 2 follow from Theorem 3.2. So let $n \geq 3$ and w be the central vertex of $\overrightarrow{K_{1,n}}$. We will prove the theorem in three cases.

Case 1: w is a source. Then, $od(w) = n$. Choose a set X of cardinality n. Let X = {1, 2, …, n}. Define $f: V \to 2^X$ by letting $f(w) = X$; $f(v_i) = X - \{i\}$; $1 \leq i \leq n$. Then, for the induced arc function we have $g_f(w, v_i) = f(w) - f(v_i) = \{i\}$. Hence, $g_f(w, v_i) = f(w) - f(v_i) = \{i\} = 1 < n - 1 = |f(v_i)|$.

Case 2: w is a sink. Then, id(w) = n. Choose a set X of cardinality n + 2. Let X = {1, 2, …., n+2}. Define f:V→ 2^X by letting f(w) = {n + 1, n +2}; $f(v_i)$ = {i}; 1 ≤ i ≤ n. Then, the induced arc function yields g_f $(v_i$, w) = $f(v_i)$ - f(w)= $f(v_i)$, since f(w) ∩ $f(v_i)$ = φ. Thus, | g_f $(v_i$, w)| = |$f(v_i)$ - f(w)|= |f(v_i)| = 1 < 2 = |f(w)|.

Case 3: od(w) > 1 and id(w) > 1. Let od(w) = r and id(w) = s. Let V_0 = {v_1, v_2, …, v_r} be the set of all vertices adjacent from w and V_1 = {u_1, u_2, …, u_r} be the set of all vertices adjacent to w. Let X_1 = {0, 1, 2, …, r} and X_2 = {r+1, r+2, …., r+s}. Choose X = X_1 U X_2 = {0, 1, 2,…, r, r+1, r+2, …, r+s} as the ground set. |X| = r + s + 1. Define f:V→ 2^X by letting f(w) = X_1; $f(v_i)$ = X_1 - {i}; 1 ≤ i ≤ r, $f(u_j)$ = {r+j}; 1 ≤ i ≤ s. The induced arc function yields g_f (w, v_i) = f(w) - $f(v_i)$ = {i}. Hence, |g_f (w, v_i)| = |f(w) - $f(v_i)$| =|{i}| = 1 < r = |$f(v_i)$|. Also, g_f ($v_{i,}$ w,) = $f(v_i)$ - f(w) = $f(v_i)$, since f(w) ∩ $f(v_i)$ = φ. Then, |g_f $(v_i$, w) | = |$f(v_i)$| = 1 < 2 = |f(w)|. Thus, in each of the three mutually exclusive and exhaustive cases, we have shown that *f* satisfies the condition in the definition of an arc-binding set-indexer and hence the result follows.

Acknowledgements

The second and third authors are thankful to the Department of Science & Technology, Government of India for supporting this research under the Project No. SR/S4/MS:277/06.

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