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Boehmians and homogeneous distributions

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Abstract

The concept of homogeneous distributions is extended to the context of a suitable Boehmian space. It is shown that every homogeneous distribution can be viewed as a convolution quotient of homogeneous functions.

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1. Introduction

The concept of a homogeneous function and that of a distribution is well-known. It is also known that any homogeneous function is determined by its values on the unit sphere. In [1], it is also shown that there exists a continuous linear isomorphism between the space $D'(S^{n-1}, \mathbb{R})$ of all distributions on the unit sphere S^{n-1} of $\mathbb{R}^n$ and the space $D'_\alpha(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ of homogeneous distributions of order α on $\mathbb{R}^n \setminus \{0\}$. (Here $\mathbb{R}$ denotes the usual real line and $\mathbb{R}^n$ denotes the n- dimensional euclidean space). In this paper we shall define the concept of a homogeneous Boehmian of order α and use this concept to identify all homogeneous distributions of order α on $\mathbb{R}^2 \setminus \{0\}$. Even though we have taken $n = 2$ to demonstrate our ideas it is possible to prove this result in the general case. We shall assume the general construction of Boehmian spaces as available in [3] and [4]. In section 2, we shall define the required concepts and obtain some preliminary results . In section 3, we shall prove that every homogeneous distributions of order α on $\mathbb{R}^2 \setminus \{0\}$ is a convolution quotient of homogeneous functions of order α on $\mathbb{R}^2 \setminus \{0\}$.

2. Test functions on the unit sphere

For $x, y \in \mathbb{R}^2$ we will write

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

and

$$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

By S^1 we denote the unit sphere in $\mathbb{R}^2$, i.e.

$$S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}.$$

Let $f : S^1 \to \mathbb{R}$. For any $\alpha \in \mathbb{R}$ we define the extension of f, of degree α, by the formula

$$(\mathcal{E}_\alpha f)(x) = |x|^\alpha f\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

In the case of $\alpha = 0$ we have

$$(\mathcal{E}_0 f)(x) = f\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

We shall freely use the spaces $C^\infty(S^1, \mathbb{R})$ (the space of test functions for distributions on the sphere S^1 equipped with a countable family of semi-norms and hence is a complete locally convex space), $D'(S^1, \mathbb{R})$ (the space of all distributions on the unit sphere), $D'_\alpha(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ (the space of all homogeneous distributions of degree α) as defined in [1]. In addition we shall let $C_\alpha^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ denote the space of all real homogeneous smooth functions of order α on $\mathbb{R}^2 \setminus \{0\}$. We shall also use the spaces $P_{2\pi}$ of periodic functions with period 2π on $\mathbb{R}$, the space $P'_{2\pi}$ of periodic distributions of period 2π and the periodic Boehmian space $\mathscr{B}_{2\pi}$ as defined in [2].

Definition 2.1. Let $f, \phi \in C^\infty(S^1, \mathbb{R}) = G$. The convolution $f * \phi$ is defined by

$$(f * \phi)(e^{ix}) = \int_0^{2\pi} f(e^{i(x-y)})\phi(e^{iy})dy$$

Definition 2.2. We can also define the convolution $v * \xi$ where $v \in D'(S^1, \mathbb{R})$, $\xi \in C^\infty(S^1, \mathbb{R})$ by $(v * \xi)(e^{ix}) = < v(y), \xi(e^{i(x-y)}) >$. (Here $<, >$ denotes the usual action in the noted variable.) It is also easy to see that this convolution extends the definition 2.1

The following lemmas are easy to verify.

Lemma 2.3. Let $f, \phi \in C^{\infty}(S^1, \mathbb{R})$. Then $f * \phi \in C^{\infty}(S^1, \mathbb{R})$.

Lemma 2.4. Let $\phi, \psi, \xi \in C^{\infty}(S^1, \mathbb{R})$. Then $\phi * \psi = \psi * \phi$, $D^k(\phi * \psi) = D^k\phi * \psi = \phi * D^k\psi$ and $(\phi * \psi) * \xi = \phi * (\psi * \xi)$

Definition 2.5. The class Δ of sequences is defined as follows:
Sequence $\{\phi_n\} \in G^N$ is a member of Δ if

1. $\displaystyle\int_0^{2\pi} \phi_n(e^{ix})dx = 1.$

2. $\displaystyle\int_0^{2\pi} |\phi_n(e^{ix})|dx \le M.$

3. support of $\phi_n = \overline{\{e^{ix} : \phi_n(e^{ix}) \neq 0\}} \to 1$ as $n \to \infty$.

The following results can be easily proved.

Lemma 2.6. If $f \in C^{\infty}(S^1, \mathbb{R})$ and $\{\phi_n\} \in \Delta$. Then $f * \phi_n \to f$ as $n \to \infty$ in $C^{\infty}(S^1, \mathbb{R})$.

Lemma 2.7. $\{\phi_n\}, \{\psi_n\} \in \Delta \Rightarrow \{\phi_n * \psi_n\} \in \Delta$.

Using the above results and the sets G and Δ one can easily form a Boehmian space in a canonical way which we call as $\mathscr{B} = B(G, \Delta)$.

As already pointed out we shall use the symbol $\mathscr{B}_{2\pi}$ for the space of periodic Boehmians as constructed in [2]. The following theorems are easy consequences of the definitions. We prefer to omit the details.

Theorem 2.8. The map $T_1 : \mathscr{B}_{2\pi} \to \mathscr{B}$ given by $T_1\left(\left[\frac{f_n}{\phi_n}\right]\right) = \left[\frac{g_n}{\xi_n}\right]$ where $g_n(e^{ix}) = f_n(x)$ and $\xi_n(e^{ix}) = \phi_n(x)$, is bijective and bi-continuous (in the delta sense).

Theorem 2.9. The map $T_2 : D'(S^1, \mathbb{R}) \to \mathscr{B}_{2\pi}$ given by $T_2(u) = \left[\frac{v * \phi_n}{\phi_n}\right]$ where $u \in D'(S^1, \mathbb{R})$, $v \in P'_{2\pi}$ is defined by $v(f) = u(g)$ for $f \in P_{2\pi}$ with $g(e^{ix}) = f(x)$ and $\{\phi_n\} \in \Delta$ is any sequence, the map T_2 is a continuous imbedding of $D'(S^1, \mathbb{R})$ into $\mathscr{B}_{2\pi}$.

It is easy to show that the space $D'(S^1, \mathbb{R})$ can be identified in a canonical way with a subset of $\mathscr{B}$. (using theorems 2.8 and 2.9.) Alternately one can use the extended convolution given by the definition 2.2 to identify the space $D'(S^1, \mathbb{R})$ using the map $v \to \left[\frac{v * \phi_n}{\phi_n}\right]$ where $\{\phi_n\}$ is any delta sequence in $\mathscr{B}$.

Definition 2.10. Let $f, \phi \in C_\alpha^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = G_1$. A multiplication of $f \circ \phi$ in $C_\alpha^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ is defined by

$$(f \circ \phi)(x) = |x|^\alpha \int_0^{2\pi} f\left(\frac{x}{|x|}\, e^{-i\xi}\right) \phi(e^{i\xi})\, d\xi$$

Definition 2.11. We can also define the multiplication $u \circ \phi$ where $u \in D'_\alpha(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$, $\phi \in C_\alpha^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ by $(u \circ \phi)(x) = \mathcal{E}_\alpha(\mathcal{R}_\alpha u * \mathcal{R}_\alpha \phi)(x)$ (here $\mathcal{E}_\alpha : D'(S^1, \mathbb{R}) \to D'_\alpha(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ and $\mathcal{R}_\alpha : D'_\alpha(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) \to D'(S^1, \mathbb{R})$ denote the isomorphisms given in [1] between the respective spaces). It is also easy to see that this multiplication extends the definition 2.10

The following lemmas are easy to verify.

Lemma 2.12. Let $f, \phi \in C_\alpha^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$. Then $f \circ \phi \in C_\alpha^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$.

Lemma 2.13. Let $\phi, \psi, \eta \in C_\alpha^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$. Then $\phi \circ \psi = \psi \circ \phi$, $D^k(\phi \circ \psi) = D^k\phi \circ \psi = \phi \circ D^k\psi$ and $(\phi \circ \psi) \circ \eta = \phi \circ (\psi \circ \eta)$

Definition 2.14. The class Δ_1 of sequences is defined as follows:
Sequence $\{\phi_n\} \in G_1^N$ is a member of Δ_1 if

1. $\displaystyle\int_0^{2\pi} \phi_n(e^{ix})dx = 1.$
2. $\displaystyle\int_0^{2\pi} |\phi_n(e^{ix})|dx \le M.$
3. Restricted support of $\phi_n = \overline{\{e^{ix} : \phi_n(e^{ix}) \neq 0\}} \to 1$ as $n \to \infty$.

The following results can be easily proved.

Lemma 2.15. If $f \in C_\alpha^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ and $\{\phi_n\} \in \Delta_1$. Then $f \circ \phi_n \to f$ as $n \to \infty$ in $C_\alpha^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$.

Lemma 2.16. $\{\phi_n\}, \{\psi_n\} \in \Delta_1 \;\Rightarrow\; \{\phi_n \circ \psi_n\} \in \Delta_1$.

Using the above results and the sets G_1 and Δ_1 one can easily form a Boehmian space which we call as $\mathscr{B}_\alpha = B(G_1, \Delta_1)$.

It is also easy to prove that the mapping $u \to \left[\dfrac{u \circ \phi_n}{\phi_n}\right]$ is a continuous imbedding of $D'_\alpha(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ into $\mathscr{B}_\alpha$.

3. Main Result

In [1], it is shown that there is a linear continuous bijection between the space $D'(S^1, \mathbb{R})$ of distribution on the unit sphere in the plane and the space $D'_\alpha(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ of homogeneous distributions of order α. These bijections are respectively given by $\mathcal{E}_\alpha$ and $\mathcal{R}_\alpha$, as mentioned in 2.11.

We are now in a position to state and prove our main result.

Theorem 3.1. The map $T : \mathscr{B} \to \mathscr{B}_\alpha$ given by $T\left(\left[\frac{f_n}{\phi_n}\right]\right) = \left[\frac{g_n}{\xi_n}\right]$ where $g_n(y) = |y|^\alpha f_n\left(\frac{y}{|y|}\right)$ and $\xi_n(y) = |y|^\alpha \phi_n\left(\frac{y}{|y|}\right)$, is bijective and bi-continuous (in the delta sense).

Proof. It is easy to see that $\mathcal{E}_\alpha(f_n)(y) = |y|^\alpha f_n\left(\frac{y}{|y|}\right) = g_n(y)$ and that $\mathcal{E}_\alpha(\phi_n)(y) = |y|^\alpha \phi_n\left(\frac{y}{|y|}\right) = \xi_n(y)$. In view of these equalities and the fact that $\mathcal{E}_\alpha$ is injective. It follows that T is injective. On the other hand if $Y = \left[\frac{g_n}{\xi_n}\right] \in \mathscr{B}_\alpha$, it is easy to see that f_n and ϕ_n defined by $f_n(e^{ix}) = (\mathcal{R}_\alpha g_n)(e^{ix}), \phi_n(e^{ix}) = (\mathcal{R}_\alpha \xi_n)(e^{ix})$ give sequences such that $X = \left[\frac{f_n}{\phi_n}\right] \in \mathscr{B}$ and that $T(X) = Y$ proving that T is surjective.

We now proceed to prove that T is continuous. Assuming the standard definition of convergence of a sequence of Boehmians, we have to prove the following: If $X_n = \left[\frac{f_{nk}}{\phi_k}\right]$, $X = \left[\frac{f_k}{\phi_k}\right]$ and $D^m(f_{nk}) \to D^m(f_k)$ as $n \to \infty$ uniformly on S^1, then $T(X_n) = \left[\frac{g_{nk}}{\xi_k}\right]$, $T(X) = \left[\frac{g_k}{\xi_k}\right]$ where $g_{nk}(y) = |y|^\alpha f_{nk}\left(\frac{y}{|y|}\right)$, $g_k(y) = |y|^\alpha f_k\left(\frac{y}{|y|}\right)$, $\xi_k(y) = |y|^\alpha \phi_k\left(\frac{y}{|y|}\right)$, $D^m(g_{nk}) \to D^m(g_k)$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}^2 \setminus \{0\}$. A simple computation using the chain rule for derivatives shows that if $D^m(f_{nk}) \to D^m(f_k)$ as $n \to \infty$ uniformly on S^1 then $D^m(g_{nk}) \to D^m(g_k)$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}^2 \setminus \{0\}$ where g_{nk} and g_k are defined as above. (Note that g_{nk} and g_k are nothing but the canonical extensions of f_{nk} and f_k to $\mathbb{R}^2 \setminus \{0\}$ and that the spherical derivatives of f_{nk} and f_k are the same as the ordinary derivatives of g_{nk} and g_k respectively (see [1])). This shows that T is continuous. Continuity of T^{-1} follows easily from the definitions. We omit the details.

Note 3.2. We know that the Boehmian space $\mathscr{B}_{2\pi}$ is larger than $P'_{2\pi}$ (see [2]). Since there is an isomorphism between the spaces $D'(S^1, \mathbb{R})$ and $P'_{2\pi}$ (given by $u \to v$ in theorem 2.9) and between $\mathscr{B}_{2\pi}$ and $\mathscr{B}$, (theorem 2.8) it is easy to see that the Boehmian space $\mathscr{B}$ is larger than $D'(S^1, \mathbb{R})$. Similarly the isomorphisms between the spaces $D'(S^1, \mathbb{R})$ and

$D'_\alpha(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ (see [1]) and between the spaces $\mathscr{B}$ and $\mathscr{B}_\alpha$ (theorem 3.1) show that the Boehmian space $\mathscr{B}_\alpha$ is larger than the space $D'_\alpha(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$. Even though we have taken $n = 2$ for simplicity, the same theorem applies for general dimensions $n \geq 3$. The main theorem also shows that every homogeneous distribution of order α (elements of $D'_\alpha(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$) can be viewed as a convolution quotient of ordinary smooth functions which are homogeneous of order α.

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