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Quadratic Spline Approximation Solution of the Generalized Nonlinear Schrödinger Equation

E.A. Al-Said and Saleh M. Hassan

Department of Mathematics, College of Science,
King Saud University, P.O. Box 2455, Riyadh 11451, KSA
E-mail: eissa-assaid@ksu.edu.sa; smhm@ksu.edu.sa

Abstract

Here we develop a finite difference method to obtain approximatesolutions of the generalized nonlinear Schrödingerequation (GNLS). The numerical method is derived through thesemidiscretization and application of the quadratic splineapproximation. Neumann boundary conditions are considered in thediscretized problem and second order difference approximation isemployed for obtaining the boundary values. Both continuous anddiscrete energy conservations are discussed and the stability ofthe present method is studied. Our investigation reveals that thepresent method is an efficient and reliable way for computing thesolitonian solutions of the GNLS equation. Two numerical examplesare provided to demonstrate the performance of our method.

Keywords: Generalized nonlinearSchrödinger equation; quadratic spline; solitary waves; stability analysis.

Introduction

It is well known that Schrödinger type equationsare commonly used in modeling the physical processes of thecomputations of nonlinear waves, pulses, and beams. In thisarticle we study an efficient numerical method for solving thegeneralized nonlinear Schrödinger(GNLS) equation

$$i\, u_t + u_{xx} + f(|u|^2)\, u = 0, \qquad |x| < \infty, \qquad t \geq 0, \tag{1.1a}$$

along with the initial condition

$$u(x,0) = \varphi(x) + i\,\psi(x), \qquad |x| < \infty \tag{1.1b}$$

where $i=\sqrt{-1}$ and $f(s)$ is sufficiently smooth with $f(0)=0$. The functions $\phi(x)$ and $\psi(x)$ are real valuedand are sufficiently smooth in the domain considered. The mostfrequently used functions f include $f(s)=s^r$ with $r>0$, $f(s)=1-e^{-s}$, $f(s)=s/1+s$, and $f(s)=\ln(1+s)$, see[1,4,5,7,8]. Equation $(1.1a)$ arises from plasma physics andquantum theory. It reduces to the nonlinearSchrödinger equation, denoted by NLS, as $f(s)=s$ [6,13]. The nonlinear term in $(1.1a)$ helps in preventingdispersion of the wave. It balances the forces of dispersion andnonlinearity in solutions. These balanced solutions representdifferent kinds of interesting solitary waves including the singlesolitary wave and collision of two or more solitons [12]. It hasbeen shown that equation $(1.1a)$ possesses, in general, aninfinite set of conservation laws [9,10]. The conservation intime of the energy can be expressed through the L_2-norm

$$\|u\|_2=\sqrt{\int_{-\infty}^{\infty}|u(x,t)|^2\,dx}=c, \qquad t>0, \tag{1.2a}$$

or the weighted L_2-norm

$$\|u\|_{2,\gamma}=\sqrt{\int_{-\infty}^{\infty}\gamma(x)|u(x,t)|^2\,dx}=c, \qquad t>0, \tag{1.2b}$$

where $\gamma(x)$ is positive and c is a constant. Conditions $(1.2a)$ or $(1.2b)$ provides an L_2-boundness of thesolution and play a critical part in the dynamics of the solitarywave models. The initially unstable Fourier modes of the wave drawenergy from the stable modes, but because of conservation, theprocess must come to an end. In fact, it is possible for theenergy to return to its initial distribution among the modes. Thisis referred to as the so-called Fermi-Pasta-Ulam recurrence[1,9,13]. Several numerical methods have been developed and usedfor solving the nonlinear and the generalized nonlinearSchrödinger equations, see for example[3,6,9-13] and the references therein. More commonly used finitedifference methods are the five classical algorithms usingsemidiscretization, moving grid adaptation, and Crank-Nicolsontype approximations [6,9,12]. In [5], several importantdifferent schemes are tested, analyzed, and compared. The use ofquartic spline approximation has been introduced in [11] wherean efficient and reliable method was developed for computinglong-time solitary wave solutions for problem (1.1). Also, in[3], a cubic spline approximation has been used to develop anumerical scheme for solving the GNLS problem (1.1).

In thispaper, we use a quadratic spline approximation for the spatialderivative to develop a numerical method for solving problem(1.1). The properties of the discrete conservation law of thepresent numerical method will be discussed under the l_2-normwhich is consistent with the original L_2-norm used forcontinuous problems. Two numerical examples will be tested in thisregard.

The numerical Method

We consider developing a numerical method for solving the GNLS problem (1.1). For the purpose of computation we may consider, as an approximation to the original problem, the following initial and boundary value problem

$$i\, u_t + u_{xx} + f(|u|^2)u = 0, \qquad a \le x \le b, \qquad 0 < t \le T, \tag{2.1a}$$

$$u(x,0) = \phi(x) + i\,\psi(x), \qquad a \le x \le b, \tag{2.1b}$$

$$u_x(a,t) = u_x(b,t) = 0, \qquad 0 < t \le T, \tag{2.1c}$$

where $|a|$ and $|b|$ are sufficiently large. Let $u(x,t) = p(x,t) + i\,q(x,t),\ a \le x \le b$ and $t > 0$, where $p(x,t)$ and $q(x,t)$ are real functions. Also let $v = [p, q]^T$ then problem (2.1) can be written as

$$v_t + A v_{xx} + g(v) = 0, \qquad a \le x \le b, \qquad 0 < t \le T, \tag{2.2a}$$

$$v(x,0) = [\phi(x) \ \ \psi(x)]^T, \qquad a \le x \le b, \tag{2.2b}$$

$$v_x(a,t) = v_x(b,t) = 0, \qquad 0 < t \le T, \tag{2.2c}$$

where

$$g(v) = f(|v|^2)\,A v \ \text{ with } A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.3}$$

Now, we discretize the space interval $[a,b]$ using the equally spaced points $x_j = a + j\,h, \quad j = 0, 1, \cdots, N+1, \ \ x_0 = a, \ \ x_{N+1} = b$, and $h = (b-a)/(N+1)$, where N is a positive integer. The spatial derivative in $(2.2a)$ is approximated by the quadratic spline collocation relation [2]

$$v_{xx}(x_{j-1},t) + 6 v_{xx}(x_j,t) + v_{xx}(x_{j+1},t) = \frac{8}{h^2}\delta_n^2 v_j + e_j, \tag{2.4}$$

where $\delta_n^2 v_j = v_{j-1} - 2v_j + v_{j+1}$, and $e_j = -\dfrac{h^2}{24} v_{xxxx}(\eta_j, t)$ for $j = 1, 2, \cdots, N$ is the error associated with this approximation and η_j lies inside a neighborhood of x_j. From equations $(2.2a)$ and (2.4) it follows that

$$(8 + \delta_n^2) w_t^j + \frac{8}{h^2} A \delta_x^2\, w_j + (8 + \delta_x^2)\, g(w_j) = 0, \ \ t > 0, \tag{2.5}$$

where $w_j = w(x_j, t)$ are approximations of $v(x_j, t)$ for $j = 1, 2, \cdots, N$. For the Neumann boundary conditions, weuse the central difference approximation $(2.2c)$ to obtain

$$\left.\begin{aligned} w(x_0 - h, t) &= w(x_1, t) + O(h^2), \qquad & w(x_{N+1} + h, t) &= w(x_N, t) + O(h^2), \\ w_t(x_0 - h, t) &= w_t(x_1, t) + O(h^2), \qquad & w_t(x_{N+1} + h, t) &= w_t(x_N, t) + O(h^2), \end{aligned}\right\} \tag{2.6}$$

where $t > 0$. Applying (2.6) for approximating the boundary values from $(2.2c)$ and (2.5) we have the second order nonlinear scheme

$$P\,w_t + \left(\frac{8}{h^2} B\,Q + P\,R\,B\right) w = 0, \qquad t > 0, \tag{2.7a}$$

$$w(0) = G, \tag{2,7b}$$

for approximating the initial and boundary value problem (2.1) where the block-tridiagonal matrices $P = \lfloor P_{ij} \rfloor$, $Q = \lfloor Q_{ij} \rfloor$, and $R = \lfloor R_{ij} \rfloor$ are defined by

$$P_{1,1} = P_{N,N} = 3\,I, \qquad P_{1,2} = P_{N,N-1} = I,$$

$$P_{j,j} = 6\,I, \qquad P_{j,j-1} = P_{j,j+1} = I, \quad j = 2, 3, \cdots, N-1,$$

$$Q_{1,1} = Q_{N,N} = -Q_{1,2} = -Q_{N,N-1} = -I,$$

$$Q_{j,j} = -2\,I, \qquad Q_{j,j-1} = Q_{j,j+1} = I, \quad j = 2, 3, \cdots, N-1,$$

$$R_{j,j} = \sigma_j\, I, \quad j = 1, 2, \cdots, N,$$

where I is the 2×2 identity matrix and $\sigma_j = f(p_j^2 + q_j^2)$, $p_j = p(x_j)$, and $q_j = q(x_j)$ for $j = 1, 2, \cdots, N$. The matrix B is the $2\,N \times 2\,N$ block-diagonal matrix $[A\ A \cdots A]$ where A is defined in equation (2.3), and the $2\,N-$dimensional vectors $w = [w_1, w_2, \cdots, w_N]^T$, with $w_j = [p_j, q_j]^T$ and $G = [g_1, g_2, \cdots, g_N]^T$ with $g_j = [\phi_j, \psi_j]^T$ where $\phi_j = \phi(x_j)$, and $\psi_j = \psi(x_j)$. It can be shown that for the conservation laws we have [11]

$$\|u\|_2 = \sqrt{\langle u, u \rangle} = c, \qquad t > 0, \tag{2,8}$$

and

$$\|u\|_{2,\Gamma} = \sqrt{\langle \Gamma\, u, u \rangle} = c, \qquad t > 0, \tag{2,9}$$

where u is a $2\,N-$dimensional vectors and Γ is a $2\,N \times 2\,N$ nonsingular and positive matrix.

Theorem 2.1

The semidiscretized problem $(2,7)$ is conservative.

Proof

Let w be the solution of problem $(2,7)$. Since P is symmetric and A is skew symmetric we have

$$\left\langle P^{-1}\, B\,Q\,w,\; w \right\rangle = 0.$$

Similarly, we find that

$$\langle R(w)Bw,\ w\rangle = w^T D_1 D_2 w = \sum_1^N \sigma_j w_j^T A w_j = 0,$$

where D_1 and D_2 are, respectively, the matrices

$$\begin{bmatrix} \sigma_1 I & 0 & . & . & . & 0 \\ 0 & \sigma_2 I & 0 & & & . \\ . & 0 & \sigma_3 I & 0 & & . \\ . & \cdots & \cdots & \cdots & \cdots & . \\ . & & & 0 & \sigma_{N-1} I & 0 \\ 0 & . & . & . & 0 & \sigma_N I \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 & . & . & . & 0 \\ 0 & A & 0 & & & . \\ . & 0 & A & 0 & & . \\ . & \cdots & \cdots & \cdots & \cdots & . \\ . & & & 0 & A & 0 \\ 0 & . & . & . & 0 & A \end{bmatrix}.$$

Observing that

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 = \langle w_t, w\rangle = \frac{8}{h^2}\langle P^{-1} BQw,\ w\rangle + \langle RBw,\ w\rangle = 0, \quad t > 0,$$

which indicate that the semidiscretized problem $(2,7)$ is conservative. □

Now, to solve the system $(2,7)$, we consider the second order implicit midpoint rule for the time integration where we have the difference formula

$$w^{(k+1)} - w^{(k)} + \frac{1}{2}\Delta t_k \left(\frac{8}{h^2} P^{-1} BQ + \frac{1}{2} R(w^{(k+1)} + w^{(k)})B\right)\left(w^{(k+1)} + w^{(k)}\right) = 0, \qquad (2.10a)$$

$$w^{(0)} = G, \qquad (2.10b)$$

where $w^{(k)}$ is an approximation to $w(t)$, and the time step $\Delta t_k = t_{k+1} - t_k$, $k \geq 0$, $0 < \Delta t_k < 1$.

Theorem 2.2
The difference scheme (2.10) is conservative.

Proof
Similar to the proof of Theorem 2.1, we first observe that

$$\left\langle P^{-1} BQ\left(w^{(k+1)} + w^{(k)}\right), \left(w^{(k+1)} + w^{(k)}\right)\right\rangle = 0,$$

and

$$\left\langle R\left(\frac{1}{2}\left(w^{(k+1)} + w^{(k)}\right)\right) B\left(w^{(k+1)} + w^{(k)}\right), \left(w^{(k+1)} + w^{(k)}\right)\right\rangle = 0.$$

Now, from $(2.10a)$ it follows that

$$\left\langle \left(w^{(k+1)} - w^{(k)}\right), \left(w^{(k+1)} + w^{(k)}\right)\right\rangle = \left\|w^{(k+1)}\right\|_2^2 - \left\|w^{(k)}\right\|_2^2 = 0.$$

Therefore, the scheme is conservative. □

Theorem 2.3
The difference formula $(2.10a)$ is unconditionally stable.

Proof
Since $|a|$ and $|b|$ can be arbitrary large, and using (2.5), we study the system derived from $(2.10a)$:

$$\left(8+\delta_x^2\right)\left(w_j^{(k+1)}-w_j^{(k)}\right)+\frac{4\Delta t_k}{h}A\delta_x^2\left(w_j^{(k+1)}+w_j^{(k)}\right)+\Delta t_k\left(8+\delta_x^2\right)g\left(\frac{1}{2}\left(w_j^{(k+1)}+w_j^{(k)}\right)\right)=0,$$
$$j=1,2,\cdots,N,\qquad k=0,1,2,\cdots, \tag{2.11}$$

Where $g(w)=f\left(p^2+q^2\right)Aw.$ Following conventional linearization process, we assume that
$$g(w)\approx f(\eta)Aw. \tag{2.12}$$

From (2.11) and (2.12) we obtain the following linearized systems of equations

$$\left(8+\delta_x^2\right)\left(w_j^{(k+1)}-w_j^{(k)}\right)+4\Delta t_k\left(\frac{1}{h^2}A\delta_x^2+f(\eta)A\left(8+\delta_x^2\right)\right)\left(w_j^{(k+1)}+w_j^{(k)}\right)=0,$$
$$j=1,2,\cdots,N,\quad k=0,1,2,\cdots. \tag{2.13}$$

Now, let $w_j^{(k)}=e^{ijh\gamma}M^k\varphi$ be thetest function, where $\gamma\in\Re$, $\varphi\in\Re^2$ and $M\in\Re^{2\times 2}$ is the amplifying matrix. Substituting thetest function into (2.13) we obtain
$$(\alpha I+\beta A)M-(\alpha I-\beta A)=0, \tag{2.14}$$

where
$$\alpha=\frac{1}{4}(3+\cos(\gamma h)),\quad \beta=\frac{\Delta t_k}{h^2}\left(\cos(\gamma h)-1+\frac{\alpha h^2}{2}f(\eta)\right). \tag{2.15}$$

Since A is skew symmetric matrix, then the matrix $\alpha I+\beta A$ is nonsingular and shares the same set ofeigenvalues with the matrix $\alpha I-\beta A$, namely, $\alpha+\beta i,\ \alpha-\beta i$. Thus, the maximal module ofthe eigenvalues of M is one. Hence, the linearized scheme isnon-dissipative and the scheme (2.10) is stable. □

Numerical results

In this section, we use the implicit finite difference method developed in section 2 to solve the following problems:

Example 3.1

The single soliton problem

$$i u_t + u_{xx} + |u|^2 u = 0, \qquad |x| < \infty, \qquad t \geq 0, \tag{3.1}$$

$$u(x,0) = \sqrt{\frac{2\alpha}{\beta}} \exp\left(\frac{i\gamma x}{2} \sec h\left(\sqrt{\alpha}\, x\right)\right), \qquad |x| < \infty, \tag{3.2}$$

Where $\alpha = \beta = \gamma = 1$.

Example 3.2

The collision of two solitons problem. Here we consider the nonlinear Schrödinger equation 3.1 along with the initial condition

$$u(x,0) = \sqrt{\frac{2\alpha}{\beta}} \left(\exp\left(\frac{i\gamma_1 x}{2}\right) \sec h\left(\sqrt{\alpha}\, x\right) + \exp\left(\frac{i\gamma_2 (x-\gamma_3)}{2}\right) \sec h\left(\sqrt{\alpha}(x-\gamma_3)\right)\right), \quad |x| < \infty, \tag{3.3}$$

Where $\alpha = 0.5$, $\beta = \gamma_1 = 1, \gamma_2 = 0.1$, and the initial location of the slower solitary wave is $\gamma_3 = 25$.

We have used our present method with a variety of $h, \Delta t_k$, a, and b values, however, for the sake of comparisonwith the numerical results given in [3,11] we give here thenumerical results for example 3.1 when $a = 30$ and $b = 70$andthose for example 3.2 as $a = 20$ and $b = 80$. Also, we choose $h = 0.5$ and $\Delta t_k = \Delta t = 0.25$ for both examples.Let n denote the time level index $t_n = n\, \Delta t$ be thecorresponding time and u_n be the numerical solution at the timelevel t_n. According to the exact solution for problem $(3.1)-(3.2)$ we have $\|u\|_2 \approx 2.82842702\; t \geq 0$ .It is observed that the total energy of the numerical solution ispreserved very well during the computations. The energy profile ofthe numerical solution u_n for problem $(3.1)-(3.2)$ are givenin Table 1. From this table it is clear that the error $\|u(t_n) - u_n\|_2$ increases linearly with time. Also, as timeincreases the computed solution for a solitary wave shifts to theright with unchanged pattern. Three-dimensional plots of thenumerical solutions along with the associated contour lines havebeen drawn. The real part p_n and the imaginary part q_n ofthe solution u_n along with their projections are plotted inFigures 1(a) and 1(b), respectively. In Figure 1(c)we plot the modules and projections of u_n at each grid point.

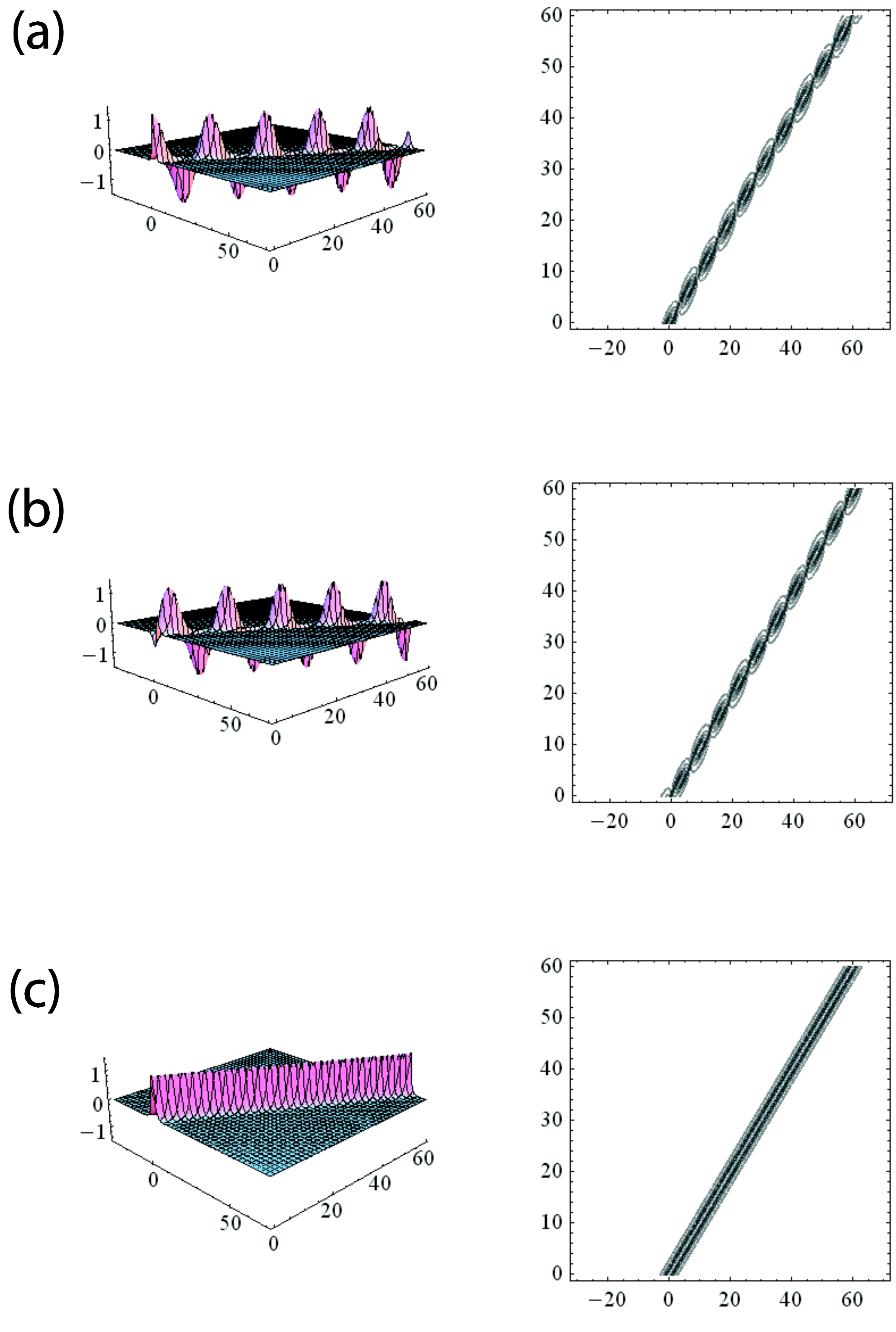

Figure 1: The computed functions (a) $p_n(x,t)$, (b) $q_n(x,t)$ and (c) $\sqrt{p_n^2(x,t)+q_n^2(x,t)}$ along with their projections for example 3.1

Table 1: The energy conservation of numerical solution of $(3.1)-(3.2)$.

n	t_n	$\|u_n\|_2$	n	t_n	$\|u_n\|_2$	n	t_n	$\|u_n\|_2$
1	0.25	2.82842742	180	45.0	2.82842795	330	82.5	2.82842788
10	2.5	2.82842742	200	50.0	2.82842826	340	85.0	2.82842789
30	7.5	2.82842742	220	55.0	2.82842821	350	87.5	2.82842795
80	20.0	2.82842787	240	60.0	2.82842805	360	90.0	2.82842789
100	25.0	2.82842798	260	65.0	2.82842816	370	92.5	2.82842793
120	30.0	2.82842794	280	70.0	2.82842875	380	95.0	2.82842807
140	35.0	2.82842803	300	75.0	2.82842848	390	97.5	2.82842783
160	37.5	2.82842797	320	80.0	2.82842801	400	100.0	2.82842752

For the second example, we use our method to solve thedifferential equation (3.1) along with the initial condition (3.3). The total energy for the exact solution of this problemis $\|u\|_2 \approx 4.75682829,\ t \geq 0$. As for the firstproblem, we observe that the total energy of the computed solutionis preserved and the error increases linearly with time. Also, astime increases both solitary waves move to the right and afterinteraction each solitary wave maintains its original shape andspeed. In Table 2, we list the energy profile of the numericalsolution u_n for this problem. The real and imaginary parts ofthe numerical solution along with their projections are plotted inFigures 2(a) and 2(b), respectively. In Figure 2(c) the energyfunction $\sqrt{p_n^2 + q_n^2}$ and the contour lines are alsoplotted for this case.

Table 2: The energy conservation of numerical solution of $(3.1)-(3.3)$.

n	t_n	$\|u_n\|_2$	n	t_n	$\|u_n\|_2$	n	t_n	$\|u_n\|_2$
2	0.5	4.75682827	70	17.5	4.75682833	140	35.0	4.75683406
10	2.5	4.75682829	80	20.0	4.75682836	150	37.5	4.75683670
20	5.0	4.75682827	90	22.5	4.75682832	160	40.0	4.75683754
30	7.5	4.75682839	100	25.0	4.75682950	170	42.5	4.75683587
40	10.0	4.75682838	110	27.5	4.75683008	180	45.0	4.75683338
50	12.5	4.75682831	120	30.0	4.75683320	190	47.5	4.75683333
60	15.0	4.75682833	130	32.5	4.75683252	200	50.0	4.75683469

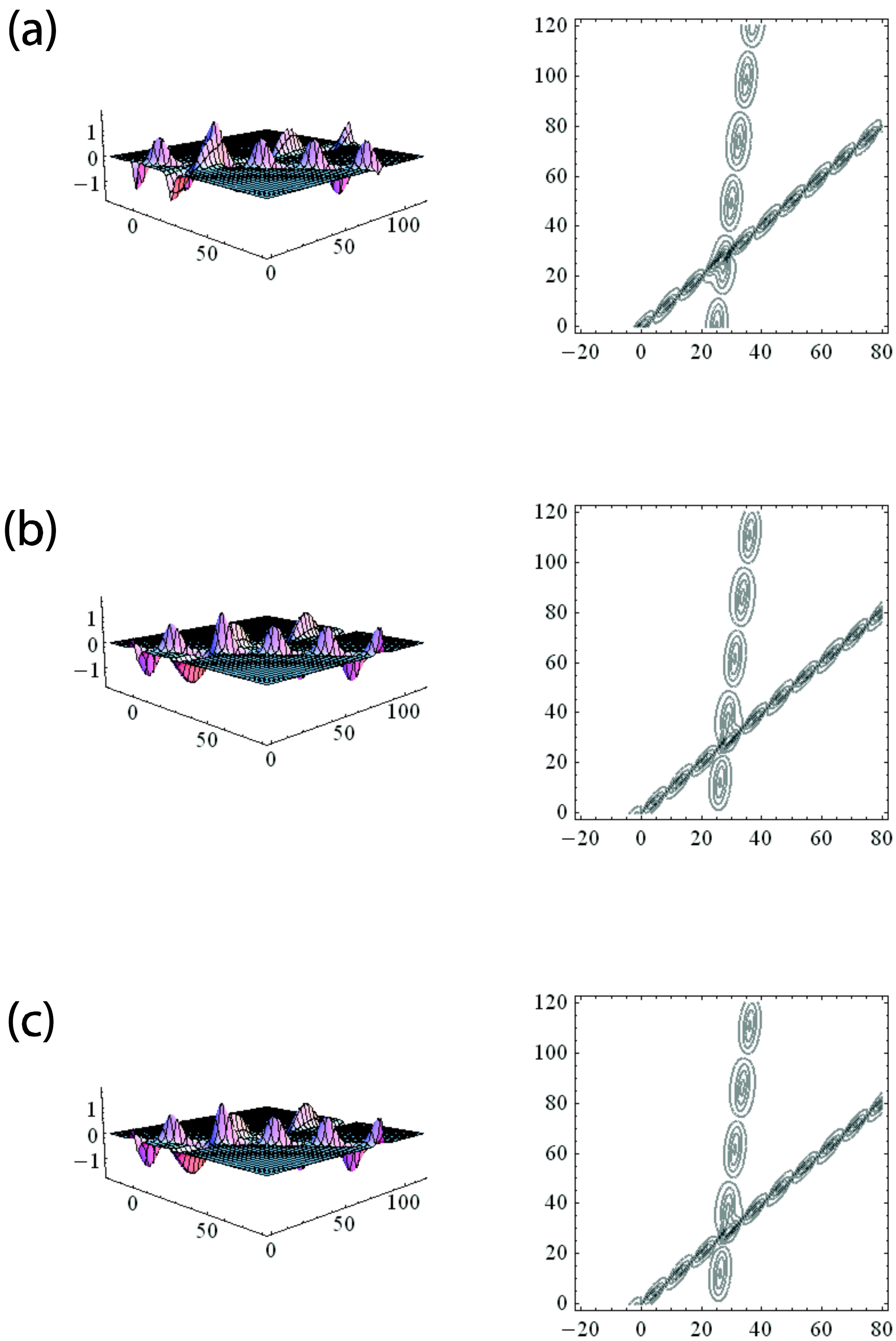

Figure 2: The computed functions (a) $p_n(x,t)$, (b) $q_n(x,t)$ and (c) $\sqrt{p_n^2(x,t)+q_n^2(x,t)}$ along with their projections for example 3.2.

Conclusion

A quadratic spline approximation for the spatial derivative hasbeen successfully used to construct a new numerical method forsolving the generalized nonlinear Schrödingerequation. The stability of the method has been studied and thenumerical experiments indicate that the l_2 −norm of solitarywave solutions remain constant for long time evaluation.

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