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On the Ratio of Exponentiated Exponential Hypergeometric Function and Gamma Random Variables

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Abstract

The present paper deals with the distribution of ratio Z = X/Y when X and Y are independently distributed as three parameter exponential distribution involving Hypergeometric function and two parameter gamma random variables respectively. The p.d.f. ,c.d.f. , moments and cumulants are also derived for the distribution

Keywords: Exponentiated exponential distribution, Hypergeometric function, Reliability, two parameter gamma distribution, moments, cumulants.

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Introduction

Joshi & Modi [5] introduced exponentiated exponential distribution with three parameters and its unique form with Hypergeometric function was given by Khan [6]. The ratio Z = X/Y has vast application in real life and it is extensively studied by many researchers like Joshi &Joshi [4],Marsaglina [7] ,Nadarajah[8] . In the present paper we derive distribution of X/Y when X and Y are independent random variables having exponentiated exponential distribution involving Hypergeometric distribution.

$$f_X(x) = \frac{\lambda\alpha\beta \quad e^{-\lambda x}\, {}_2F_1\left(1-\alpha\ ,\beta_1;\gamma\,;\beta e^{-\lambda x}\right)}{1-{}_2F_1\left(-\alpha\ ,\beta_1;\gamma\,;\beta\ \right)} \tag{1.1}$$

Where

$\gamma > 0, |\beta e^{-\lambda x}| < 1, x > 0, \alpha > 0, \lambda > 0, |\beta| < 1.$

and two parameter gamma function with p.d.f.given as-

$$f_Y(y) = (a^m / \Gamma m) e^{-ay} y^{m-1}; \tag{1.2}$$

for

m>0, a>0, y>0.

P.d.f. and c.d.f. Ratio of X & Y.

Theorem 2.1: If X and Y are distributed according to equations (1.1) and (1.2) respectively then c.d.f. and p.d.f. of Z=X/Y can be given as -

$$F_z(z) = \frac{\alpha\beta}{1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n(\beta_1)_n}{(\gamma)_n(n+1)!} \frac{\beta^n\left(a^m - [\lambda z(1+n)+a]^m\right)}{[\lambda z(1+n)+a]^m} \tag{2.1}$$

and

$$f_z(z) = \frac{\lambda\alpha\beta}{1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n(\beta_1)_n}{(\gamma)_n(n)!} \frac{\beta^n a^m(-m)}{[\lambda z(1+n)+a]^{m+1}} \tag{2.2}$$

where; $x > 0, \alpha > 0, \beta_1 > 0, \lambda > 0, 0 < \beta < 1.$

Proof: Lets X/Y = Z then its c.d.f. can be defined as

$$F_z(z) = P(\frac{X}{Y} \le Z) = P(X \le YZ)$$

$$= \int_0^{\infty} F_X(yz) f_y(y) dy$$

$$= (a^m / \Gamma m) \int_0^{\infty} \frac{\alpha\beta}{1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n(\beta_1)_n\beta^n}{(\gamma)_n(n+1)!} \left(e^{-\lambda yz(1+n)} - 1\right) e^{-ay} y^{m-1} dy.$$

where F $_X$ (yz) is obtained by using (Eq.(1.2) page 252, khan[6])and finally by using well known formula for gamma function

$$\int_0^{\infty} e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n} \; ; \text{ n } > 0$$

$$F_z(z) = \frac{\alpha\beta}{1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n(\beta_1)_n}{(\gamma)_n(n+1)!} \frac{\beta^n\left(a^m - [\lambda z(1+n)+a]^m\right)}{[\lambda z(1+n)+a]^m}$$

and differentiation above equation w.r.t z we can easily obtained

$$f_z(z) = \frac{\lambda\alpha\beta}{1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n(\beta_1)_n}{(\gamma)_n(n)!} \frac{\beta^n a^m(-m)}{[\lambda z(1+n)+a]^{m+1}}$$

Hazard Rate Function

The hazard rate function defined by h (x) = $\frac{f(x)}{1-F(x)}$ and for the variate z it is given as

$$h_z(z)=\frac{\frac{\lambda\alpha\beta}{1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)}\sum_{n=0}^{\infty}\frac{(1-\alpha)_n(\beta_1)_n}{(\gamma)_n(n)!}\frac{\beta^n a^m(-m)}{[\lambda z(1+n)+a]^{m+1}}}{1-\frac{\alpha\beta}{1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)}\sum_{n=0}^{\infty}\frac{(1-\alpha)_n(\beta_1)_n}{(\gamma)_n(n+1)!}\frac{\beta^n\left(a^m-[\lambda z(1+n)+a]^m\right)}{[\lambda z(1+n)+a]^m}} \tag{3.1}$$

Further

$$h_z(o)=\frac{\lambda\alpha\beta\ (-m)\ {}_2F_1(1-\alpha,\beta_1;\gamma;\beta)}{a[1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)]} \tag{3.2}$$

and $h(\infty)=0$, the hazard rate function ranges from h(0) to zero.

The Survival or Reliability function for c.d.f. of equation (1.2) is given as:

S (z) =1- F_z(z)

$$S(z)==1-\frac{\alpha\beta}{1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)}\sum_{n=0}^{\infty}\frac{(1-\alpha)_n(\beta_1)_n}{(\gamma)_n(n+1)!}\frac{\beta^n\left(a^m-[\lambda z(1+n)+a]^m\right)}{[\lambda z(1+n)+a]^m}\text{........}(3.3)$$

Moments

The rth moments for random variable Z=X/Y whose p.d.f. is given by equation (2.2) is -

$$E(z^r)=\frac{\lambda\alpha\beta}{1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)}\sum_{n=0}^{\infty}\frac{(1-\alpha)_n(\beta_1)_n}{(\gamma)_n(n)!}\frac{\beta^n a^m(-m)}{}$$

$$\int_0^{\infty}[\lambda z(1+n)+a]^{-m-1}z^r\,dz$$

$$=\frac{\alpha\beta}{1-{}_2F_1(-\alpha,\beta_1;\gamma;\beta)}\sum_{n=0}^{\infty}\frac{(1-\alpha)_n(\beta_1)_n}{(\gamma)_n(n)!}\frac{\beta^n(-m)}{}$$

$$\frac{a^r}{\lambda^r(1+n)^{r+1}}\mathbf{B}(m-r,r+1) \tag{4.1}$$

Where

$$B(p,q)=\frac{\Gamma p\Gamma q}{\Gamma p+q}.$$

On taking r=1, 2, 3, 4. (m > r), we can easily obtained first, second, third &fourth moments about origin.

Relation between Cumulants and Moments

Since cumulants is defined as -

$$K(t) = \log_e M(t) = k_1 t + k_2 \frac{t^2}{2\,!} + k_3 \frac{t^3}{3\,!} + ...k_r \frac{t^r}{r\,!} + ... \tag{5.1}$$

The coefficients $k_1, k_2, ...k_r$ are called the first, second….. rth cumulants. Further relation between moments and cumulants is given as.

$$\mu'_r = \sum_{j=1}^{r} {}^{r-1}C_{j-1} \mu'_{r-j} K_j . \tag{5.2}$$

By using (4.1), we have

$$\frac{\Gamma(r+1)}{\lambda^r} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n (\beta_1)_n}{(\gamma)_n (n)!} \frac{\beta^{n+1}}{(n+1)^{r+1}} .$$

$$\frac{\alpha\beta}{1-{}_2F_1(-\alpha, \beta_1; \gamma; \beta)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n (\beta_1)_n}{(\gamma)_n (n)!} \frac{\beta^n (-m)}{} \frac{a^r}{\lambda^r (1+n)^{r+1}} \mathbf{B}(m-r, r+1)$$

$$= \sum_{j=1}^{r} {}^{r-1}C_{j-1} \frac{\alpha\beta}{1-{}_2F_1(-\alpha, \beta_1; \gamma; \beta)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n (\beta_1)_n}{(\gamma)_n (n)!} \frac{\beta^n (-m)}{}$$

$$\frac{a^{r-j}}{\lambda^{r-j} (1+n)^{r-j+1}} \mathbf{B}(m-r+j, r-j+1) K_j \tag{5.3}$$

On specializing j=1, 2...r, we can get the values of $k_1, k_2, ...k_r$.

Particular Cases

Case I: Setting $\beta_1 = \gamma$, a=1, $\lambda = 1$, the p.d.f. & c.d.f. of z is given by

$$f_z(z) = \frac{\alpha\beta}{1-(1-\beta)^\alpha} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n \beta^n (-m)}{n!} [z(1+n)+1]^{-m-1} \tag{6.1}$$

$$F_z(z) = \frac{\alpha\beta}{1-(1-\beta)^\alpha} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n \beta^n}{(n+1)!} \{\frac{1-[z(1+n)+1]^m}{[z(1+n)+1]^m}\} \tag{6.2}$$

Case II: Setting $\beta = 1$, n=0 and $\alpha = -1$ in above equations we get

$$f_z(z) = \frac{m}{(z+1)^{m+1}} \tag{6.3}$$

$$F_z(z) = 1 - \frac{1}{(z+1)^m} \tag{6.4}$$

This is known result given by Joshi and Joshi [4, eq. (5.3) & (5.4) page 73]

Case III: Setting m=1 in (6.3) & (6.4)

$$f_z(z) = \frac{1}{(z+1)^2} \tag{6.5}$$

$$F_z(z) = \frac{z}{(z+1)} \tag{6.6}$$

Which is also a known result given by Joshi and Joshi [4, eq. (5.5) & (5.6) page 74]

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